Diffusion with Stochastic Resetting

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Introduction: Search Problems

Search Problems are ubiquitous in nature and occur in a variety of contexts

- from foraging of animals to target location on DNA
- from internet searches to the mundane task of finding one's misplaced possessions

How does one search for lost keys?

after a while go back to where they should be and start looking again i.e. reset the search

Plan: Diffusion with Stochastic Resetting

Plan

I. Recap of diffusion equation, absorbing target, mean first passage time

II. Stochastic resetting

III. Many searchers

References:

M. R. Evans and S. N. Majumdar, Phys. Rev. Lett. **106**, 160601 (2011)

M. R. Evans and S. N. Majumdar, J. Phys. A: Math. Theor. **44**, 435001 (2011)

I Diffusion Equation (1d)

Forward Equation for $p(x, t|x_0)$, the probability density of the position x of a diffusing particle at time t, having begun from x_0 at time 0

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x^2}$$

with initial condition $p(x, 0|x_0) = \delta(x - x_0)$.

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Solution (initial value Green function)

$$p(x,t|x_0) = \frac{1}{\sqrt{4Dt}}\exp-\frac{(x-x_0)^2}{4Dt}$$

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Solution (initial value Green function)

$$p(x,t|x_0) = \frac{1}{\sqrt{4Dt}}\exp - \frac{(x - x_0)^2}{4Dt}$$

Also satisfies Backward Equation

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x_0^2}$$

Absorbing target at the origin

Boundary condition $p(x, t|x_0) = 0$ when x or $x_0 = 0$

Survival probability $q(t|x_0) = \int_0^\infty \mathrm{d}x\, p(x, t|x_0)$ satisfies

Backward equation

$$\frac{\partial q(t|x_0)}{\partial t} = D\frac{\partial^2 q(t|x_0)}{\partial x_0^2}$$

with boundary/initial conditions $q(t|0) = 0$ and $q(0|x_0) = 1$ $x_0 \neq 0$

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Solution

$$q(t|x_0) = \text{erf}\left(\frac{x_0}{2\sqrt{Dt}}\right) \simeq \frac{x_0}{\sqrt{D\pi}t^{1/2}} \quad \text{for } t \gg 1$$

The mean first passage time $T = -\int_0^\infty \mathrm{d}t\, t\frac{\partial q(t|x_0)}{\partial t} \to \infty$

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Conclusion

a purely diffusive search for a target is not efficient since mean time to absorption diverges

II Diffusion with resetting

Now consider resetting the particle to the initial position x_0 with rate r:

Forward equation for $p(x, t|x_0)$ now reads (no absorbing target)

$$\frac{\partial p(x,t|x_0)}{\partial t} = D\frac{\partial^2 p(x,t|x_0)}{\partial x^2} - rp(x,t|x_0) + r\delta(x-x_0)$$

i.e. loss rate r from all $x \neq x_0$ provides source at x_0

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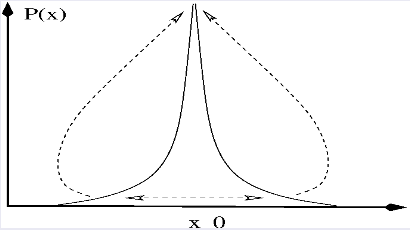
For $t \to \infty$ the stationary state probability density is

$$p_{\rm st}(x|x_0) = \frac{\alpha_0}{2}\exp(-\alpha_0|x-x_0|) \quad \text{where} \quad \alpha_0 = \sqrt{r/D}$$

Nonequilibrium stationary state



Survival probability with resetting

The Backward equation for the survival probability $q(t|z)$ when there is an absorbing target at the origin reads

$$\frac{\partial q(t|z)}{\partial t} = D\frac{\partial^2 q(t|z)}{\partial z^2} - rq(t|z) + rq(t|x_0)$$

with boundary/initial conditions $q(t|0) = 0$ and $q(0|z) = 1$ $z \neq 0$

z is the starting position (variable) and x_0 is the *fixed* resetting position

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Laplace transform satisfies

$$D\frac{\mathrm{d}^2 \widetilde{q}(s|z)}{\mathrm{d}z^2} - (s+r)\widetilde{q}(s|z) = -1 - r\widetilde{q}(s|x_0)$$

Solution which fits boundary/initial conditions

$$\widetilde{q}(s|z) = \left[1 + r\widetilde{q}(s|x_0)\right]\frac{1 - \mathrm{e}^{-\alpha z}}{s+r}$$

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Solution which fits boundary/initial conditions

$$\widetilde{q}(s|z) = \left[1 + r\widetilde{q}(s|x_0)\right]\frac{1-\mathrm{e}^{-\alpha z}}{s+r}$$

Then solve self-consistently for

$$\boxed{\widetilde{q}(s|x_0) = \frac{1-\mathrm{e}^{-\alpha x_0}}{s+r\mathrm{e}^{-\alpha x_0}}} \quad \text{where} \quad \alpha = \left(\frac{s+r}{D}\right)^{1/2}$$

Mean first passage time (MFPT)

MFPT $T = -\int_0^\infty \mathrm{d}t\, t \frac{\partial q(t|x_0)}{\partial t} = \widetilde{q}(0|x_0)$ is now finite for $0 < r < \infty$

$$T = \frac{\mathrm{e}^{x_0 (r/D)^{1/2}} - 1}{r}$$

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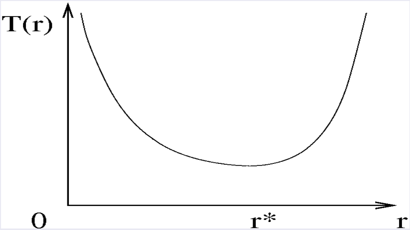
$T(r)$ has a minimum at r^*



$$\frac{\mathrm{d}T}{\mathrm{d}r} = 0$$

$$\Rightarrow \frac{y}{2} = 1 - \mathrm{e}^{-y}$$

where $y = x_0 (r/D)^{1/2}$

y= distance from target : typical distance diffused between resets

Optimal $y^* = 1.5936...$

Survival probability

The long-time behaviour of the $q(t|x_0)$ is now controlled by

simple pole of $\widetilde{q}(s|x_0) = \dfrac{1 - \mathrm{e}^{-\alpha x_0}}{s + r\mathrm{e}^{-\alpha x_0}}$ at

$$s_0 = -r \exp -x_0[(r + s_0)/D]^{1/2}$$

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For $y = x_0(r/D)^{1/2} \gg 1 \quad s_0 \simeq -r \exp -y$ and

$$q(t|x_0) \simeq \exp(-rt\,\mathrm{e}^{-y})$$

which has the form of a Gumbel distribution which gives the cumulative distribution for the maximum of independent r.v.s

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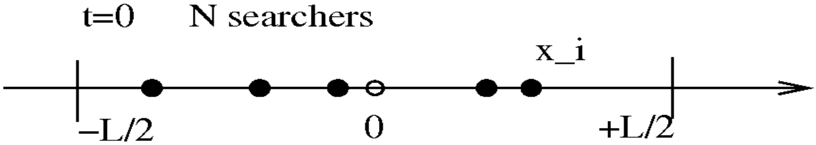
Explanation

On average there are rt resets. For each reset the process is "renewed" and the particle trajectory is independent. The particle must not reach the origin in any reset to survive.

So survival is probability that max excursion to left, out of $\simeq rt$ resets, is less than x_0

III Many Searchers

Consider the survival probability of a target at the origin in the presence of many particles (searchers/traps).



N searchers beginning at $x_i \quad i = 1 \dots N$

$p(x) = \frac{1}{L} \quad |x| \leq \frac{L}{2}$ (uniform distribution) density $\quad \rho = \frac{N}{L}$

Survival probability of target $\boxed{Q(t|\{x_i\}) = \prod_{i=1}^{N} q(t|x_i)}$

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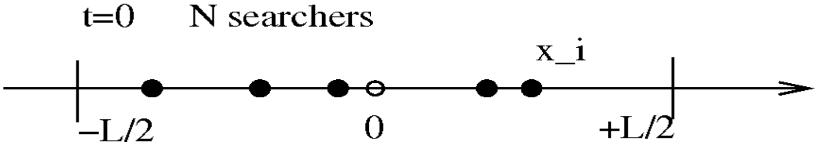

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For diffusive particles

decay with time t is $Q(t) \sim \exp(-\lambda\rho\sqrt{Dt})$ where λ is a constant which depends on whether we consider the average or typical behaviour

How does resetting affect this result?

Many Searchers: General Formulation

Average (annealed) probability

$$
\begin{aligned}
Q^{av}(t) &= \prod_{i=1}^{N} \langle q(t|x_i)\rangle_{x_i} = \exp N \ln \langle q(t|x)\rangle_x \\
&\to \exp -2\rho \int_0^\infty \mathrm{d}x\,(1 - q(t|x))
\end{aligned}
$$

Typical (quenched) probability

$$
\begin{aligned}
Q^{typ}(t) &= \exp\langle \ln Q(t|\{x_i\})\rangle_{\{x_i\}} = \exp N \langle \ln q(t|x)\rangle_x \\
&\to \exp 2\rho \int_0^\infty \mathrm{d}x\ \ln q(t|x)
\end{aligned}
$$

Many Searchers: Results

Diffusive case recall $q(t|x) = \mathrm{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$ which yields

$$Q^{av,typ}(t) = \exp\left[-\lambda^{av,typ}\rho\sqrt{Dt}\right]$$

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With Resetting

$rt \ll 1$ recovers diffusive results

$rt \gg 1$

$$Q^{av}(t) \simeq \text{constant}\; t^{-2\rho\sqrt{D/r}}$$

$$Q^{typ}(t) \simeq \exp\left[-t\,\rho\sqrt{Dr}\,8(1-\ln 2) + O(t^{1/2})\right]$$

Explanation of different behaviours

$Q^{av}(t) \gg Q^{typ}(t)$ since average behaviour dominated by rare realisations of $\{x_i\}$ far from target $\rightarrow$ **memory of initial conditions**

Summary and Outlook

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- Resetting gives finite mean first passage time
- Survival probability for single searcher decays exponentially
- Connection to statistics of extremes and a renewal process
- For many searchers survival probability of target *typically* decays exponentially but rare events make *average* decay more slowly

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Outlook

- Higher spatial dimensions can be studied
- $r(x)$ can be made position dependent
- A target distribution can be considered
- A resetting distribution $\mathcal{P}(x_0)$ can be considered
- Other optimisation problems e.g. cost for resetting