Emergence of long range order in the XY model

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Outline

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The model

XY Hamiltonian :

$$\mathcal{H} = \sum \frac{p_i^2}{2} + \frac{J}{2k} \sum_{i,j} \epsilon_{i,j}[1 - cos(\theta_i - \theta_j)]$$

where:

$$\{p_i; \theta_i\} \in \Omega = [-\infty; \infty]^N \times (0; 2\pi]^N$$

$$\epsilon_{i,j} = \begin{cases} 1 & i,j \;\; connected \\ 0 & i,j \;\; otherwise \end{cases}$$

$$k = \frac{\sum_{i>j} \epsilon_{i,j}}{N}$$

$$J > 0 \;\; ferromagnetic$$

More in detail

Adiacency matrix $\epsilon_{i,j}$

Symmetric matrix which encodes the information about the spins connections.

$$\epsilon_{i,j} = \begin{cases} 1 & i,j \ connected \\ 0 & i,j \ \ otherwise \end{cases} \Rightarrow Ex: \ \ full\ coupling = \begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix}$$

Degree of a vertex k

Number of connections per spin.

$$\boxed{k = \sum_j \epsilon_{i,j} = \frac{\sum_{i>j} \epsilon_{i,j}}{N}}$$

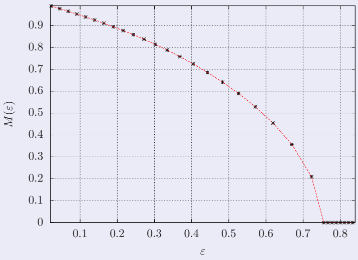
What if the spins are fully coupled?

$\epsilon_{i,j} = 1 \implies$ Hamiltonian Mean Field Model[1]

$$\mathcal{H} = \sum \frac{p_i^2}{2} + \frac{J}{2N} \sum_{i,j} [1 - cos(\theta_i - \theta_j)]$$

We define a global order parameter: Magnetisation

$$\begin{cases} m_x = \frac{1}{N} \sum cos\theta_i \\ m_y = \frac{1}{N} \sum sin\theta_i \end{cases} \implies \left|\vec{M}\right| = \sqrt{m_x^2 + m_y^2}$$



[1] Antoni, Ruffo, Phys. Rev. E Vol.52, 2361-2374

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From nearest neighbours coupling to the full coupling regime

We define a dilution parameter γ

- Links per spin: $k = 2^{2-\gamma} N^{\gamma-1}$
  - $1 \leq \gamma \leq 2$

$$\begin{matrix} 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{matrix} \Longrightarrow \begin{matrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{matrix}$$

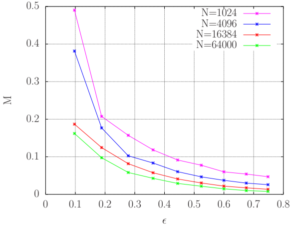
$\gamma = 2 \Rightarrow k = N$ $\qquad$ $\gamma = 1 \Rightarrow k = 2$

full coupling $\Rightarrow$ *nearest neighbours coupling*

Low dilution values: approaching the 1-D topology

For $\gamma < 1.5$:

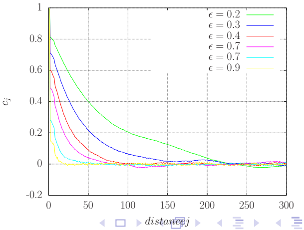
We integrated numerically the Hamilton equations for the dynamics and we considered the equilibrium magnetisation $\left\langle \left| \vec{M} \right| \right\rangle$.



The residual magnetisation vanishes with size $\Rightarrow$No phase transition of second order type

To control the eventual presence of a Kosterlitz-Thouless phase transition, we considered the correlation function:

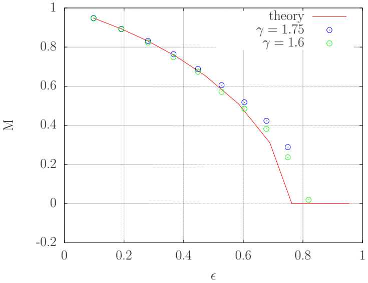
$$c_j = \frac{1}{N} \sum_{i=1}^{N} \cos(\theta_i - \theta_{(i+j) \mod N})$$



High dilution values: the mean field phase transition

For $\gamma > 1.5$:

The equilibrium magnetisation $\left\langle \left| \vec{M} \right| \right\rangle$ recovers the mean field phase transition in the thermodynamic limit.



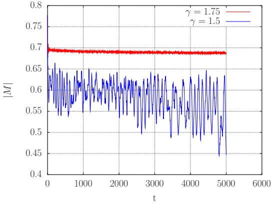
To ensure the reaching of the equilibrium state, we controlled the scaling of the magnetisation variance:

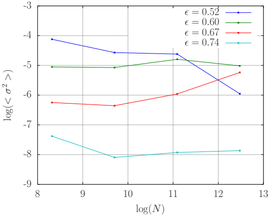
$$\sigma^2 = \left\langle M^2 - \langle M \rangle^2 \right\rangle \sim \frac{1}{N}$$

What for $\gamma_c = 1.5$?

$\gamma_c = 1.5$

Critical value for the passage between short and low range regimes.





For $\gamma_c = 1.5$ it exists an energy range $0.4 \lesssim \epsilon \leq 0.75$ in which:

The magnetisation shows important fluctuations.

It doesn't reach the equilibrium on the timescales considered.

These effects are size independent.

The low temperatures approximation

We considered an approximated Hamiltonian:

$$\cos(\theta_i - \theta_j) \approx 1 - \frac{(\theta_i - \theta_j)^2}{2}$$

$$\Rightarrow H \approx \sum_i \frac{p_i^2}{2} + \frac{J}{4k} \sum_{i,j} \epsilon_{i,j} (\theta_i - \theta_j)^2$$

Since at equilibrium $\{\theta_i,\ p_i\}$ are Gaussian distributed variables:

$$\theta_i = \sum_{k=1}^{N} \alpha_k(t) \cos(\frac{2\pi ki}{N} + \phi_k) \Longrightarrow$$

$$p_i = \sum_{k=1}^{N} \dot{\alpha}_k(t) \cos(\frac{2\pi ki}{N} + \phi_k)$$

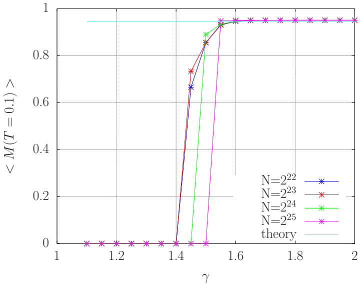
where ϕ_k are randomly distributed phases on the circle.

An approximation for the magnetisation

Injecting the waves representation in the Hamiltonian and averaging on the random phases:

$$\ddot{\alpha_k} = -\omega_k^2 \alpha_k = -(1 - \lambda_k)\alpha_k$$

where $\{\lambda_k\}$, $k \in [1, N]$ are the eigenvalues of the matrix $\epsilon_{i,j}$



ω_k and α_k are related by the equipartition of energy:

$$\alpha_k^2 \omega_k^2 \approx \frac{2T}{N}$$

Hence for the magnetisation[a] $\left\langle \left| \vec{M} \right| \right\rangle$ in the low temperatures regime:

$$\log(\left\langle \left| \vec{M} \right| \right\rangle) \approx -\sum_k \frac{\alpha_k^2}{4} \Rightarrow$$

$$\left\langle \left| \vec{M} \right| \right\rangle \approx \exp(-\frac{T}{2N} \sum_k \frac{1}{1-\lambda_k})$$

[a] Leoncini, Verga, Ruffo Phys.Rev.E 57 (1998)

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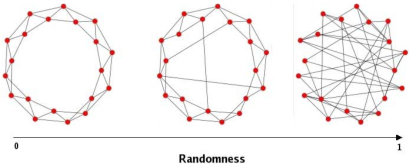
A way to control the randomness

Having considered the model on the regular lattice topology, we introduce now a controlled amount of randomness in the network.

Watts-Strogatz model[2]:

We rewired the links according to a rewiring probability p



p=0 – – – – – – – – – - -→p=1

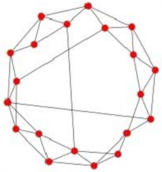
regular lattice→random network

[2] Watts, Strogatz, Nature 393 (1998), 440–442.

The Small World network

It exist an interval for p in which the network has:

- short average path length: rewiring introduces shortcuts
- high clustering



In this regime, the network is a Small World network.

Randomness induces the phase transition

Two limit cases:

- $p \approx 0 \ \cup \ \gamma < 1.5$: regular lattice configuration$\Rightarrow$no phase transition
- $p = 1 \ \cup \ \gamma \in (1; 2]$: random network configuration $\Rightarrow$mean field phase transition recovered in TD limit[3]

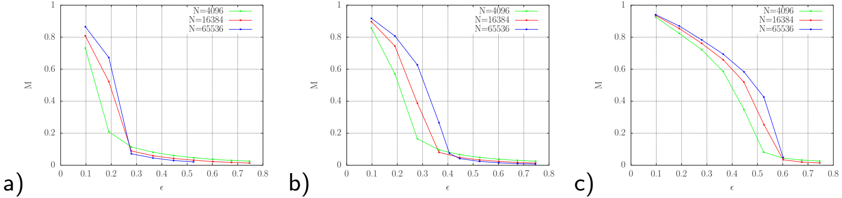
What about the Small World regime?

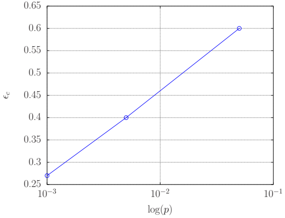
The mean field phase transition arises for intermediate values of p $\left(p \geq \frac{1}{N}\right)$, even for low γ.

[3] Ciani et al, Nonlinear Physical Science, 2011, Vol.0, 83-132

The dependence of ϵ_c on the rewiring probability

Moreover the transition energy ϵ_c depends on p:





$\Rightarrow$ $\epsilon_c \sim \log(p)$

Figure: $\gamma = 1.25$, $a)$ $p = 0.001$, $b)$ $p = 0.005$, $c)$ $p = 0.05$

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Conclusions...

- We studied the XY model for various dilutions on a regular lattice: for low dilutions it doesn't undergo a phase transition, while, in the high dilution regime, the mean field transition of the magnetisation arises.
- The XY model on a regular lattice shows a non trivial behaviour when the dilution overcomes the threshold $\gamma = 1.5$.
- Considering the complex network, the mean field phase transition is recovered even for low dilution values when the network is in the Small World regime ($p > \frac{1}{N}$)
- The transition energy ϵ_c varies in correspondence to the randomness: $\epsilon_c \sim \log(p)$

....and perspectives...

Further developments :

- Analytic proof of critical point for the dilution $\gamma = 1.5$.
- Analysis of the interplay between γ and the rewiring probability p.
- Deeper understanding of the mechanism underlying the logarithmic dependence of ϵ_c.