An introduction to particle rare event simulation

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INRIA Bordeaux- Sud Ouest & IMB & CMAP

Computation of transition trajectories and rare events in non equilibrium systems , ENS Lyon, June 2012

Some hyper-refs

- Feynman-Kac formulae, Genealogical & Interacting Particle Systems with appl., Springer (2004)
- Sequential Monte Carlo Samplers JRSS B. (2006). (joint work with A. Doucet & A. Jasra)
- A Backward Particle Interpretation of Feynman-Kac Formulae M2AN (2010). (joint work with A. Doucet & S.S. Singh)
- On the concentration of interacting processes. Foundations & Trends in Machine Learning [170p.] (2012). (joint work with Peng Hu & Liming Wu) [+ Refs]
- More references on the website : Feynman-Kac models and particle systems [+ Links]

Basic notation

$\mathcal{P}(E)$ probability meas., $\mathcal{B}(E)$ bounded functions on E.

- $(\mu, f) \in \mathcal{P}(E) \times \mathcal{B}(E) \quad \longrightarrow \quad \mu(f) = \int \mu(dx)\ f(x)$
- $Q(x_1, dx_2)$ **integral operators** $x_1 \in E_1 \rightsquigarrow x_2 \in E_2$

$$\begin{aligned} Q(f)(x_1) &= \int Q(x_1, dx_2) f(x_2) \\ [\mu Q](dx_2) &= \int \mu(dx_1) Q(x_1, dx_2) \qquad (\Longrightarrow [\mu Q](f) = \mu[Q(f)]\ ) \end{aligned}$$

- **Boltzmann-Gibbs transformation**

  [Positive and bounded potential function G]

$$\mu(dx) \quad \mapsto \quad \Psi_G(\mu)(dx) = \frac{1}{\mu(G)}\ G(x)\ \mu(dx)$$

Importance sampling and optimal twisted measures

$\mathbb{P}(X \in A) = \mathbb{P}_X(A) = 10^{-10} \rightsquigarrow$ Find $\mathbb{P}_Y$ t.q. $\mathbb{P}_Y(A) = \mathbb{P}(Y \in A) \simeq 1$

$\rightsquigarrow$ **Crude Monte Carlo sampling Y^i i.i.d. $\mathbb{P}_Y$**

$$\mathbb{P}_Y\left(\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}\, 1_A\right) = \mathbb{P}_X(A) \simeq \mathbb{P}_X^N(A) := \frac{1}{N}\sum_{1\leq i\leq N}\frac{d\mathbb{P}_X}{d\mathbb{P}_Y}(Y^i)\; 1_A(Y^i)$$

Optimal twisted measure = Conditional distribution

$$\text{Variance} = 0 \Longleftrightarrow \mathbb{P}_Y = \Psi_{1_A}(\mathbb{P}_X) = \text{Law}\,(X \mid X \in A)$$

$$\Downarrow$$

Perfect or MCMC samplers =acceptance-rejection techniques

BUT

Very often with very small acceptance rates

Conditional distributions and Feynman-Kac models

Example : Markov chain models X_n restricted to subsets A_n

$$\mathbf{X} = (X_0, \ldots, X_n) \in \mathbf{A} = (A_0 \times \ldots \times A_n)$$

Conditional distributions

$$\mathrm{Law}\,(\mathbf{X} \mid \mathbf{X} \in \mathbf{A}) = \mathrm{Law}((X_0, \ldots, X_n) \mid X_p \in A_p, \quad p < n) = \mathbb{Q}_n$$

and

$$\mathrm{Proba}(X_p \in A_p, \quad p < n) = \mathcal{Z}_n$$

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given by the Feynman-Kac measures

$$d\mathbb{Q}_n := \frac{1}{\mathcal{Z}_n} \left\{ \prod_{0 \leq p < n} G_p(X_p) \right\} d\mathbb{P}_n$$

with

$$\mathbb{P}_n = \mathrm{Law}(X_0, \ldots, X_n) \quad \text{and} \quad G_p = 1_{A_p}, \quad p < n$$

Feynman-Kac models (general $G_n(X_n)$ & $X_n \in E_n$)

Flow of n-marginals

$$\eta_n(f) = \gamma_n(f)/\gamma_n(1) \quad \text{with} \quad \gamma_n(f) := \mathbb{E}\left(f(X_n) \prod_{0 \le p < n} G_p(X_p) \right)$$

$$\Downarrow (\gamma_n(1) = \mathcal{Z}_n)$$

Nonlinear evolution equation :

$$\begin{array}{rcl} \eta_{n+1} & = & \Psi_{G_n}(\eta_n) M_{n+1} \\ \mathcal{Z}_{n+1} & = & \eta_n(G_n) \times \mathcal{Z}_n \end{array}$$

with the Markov transitions

$$M_{n+1}(x_n, dx_{n+1}) = \mathbb{P}(X_{n+1} \in dx_{n+1} \mid X_n = x_n)$$

Note : $[X_n = (X'_0, \ldots, X'_n) \ \& \ G_n(X_n) = G'(X'_n)] \Longrightarrow \eta_n = \mathbb{Q}'_n$

Interacting particle samplers

Nonlinear evolution equation :

$$\begin{aligned}\eta_{n+1} &= \Psi_{G_n}(\eta_n)M_{n+1}\\ \mathcal{Z}_{n+1} &= \eta_n(G_n)\times\mathcal{Z}_n\end{aligned}$$

$\rightsquigarrow$ **Sequential particle simulation technique**

$$M_n\text{-propositions} \oplus G_n\text{-acceptance-rejection with recycling}$$

$$\Updownarrow$$

$\rightsquigarrow$ **Genetic type branching particle model**

$$\xi_n = (\xi_n^i)_{1\leq i\leq N} \xrightarrow{G_n-selection} \widehat{\xi}_n = (\widehat{\xi}_n^i)_{1\leq i\leq N} \xrightarrow{M_n-mutation} \xi_{n+1} = (\xi_{n+1}^i)_{1\leq i\leq N}$$

Note :

$[X_n = (X_0', \ldots, X_n')\ \&\ G_n(X_n) = G'(X_n')] \Longrightarrow$ **Genealogical tree model**

$\supset$ *Continuous time models* $\supset$ *Langevin diffusions*

$$X_n := X'_{[t_n, t_{n+1}[} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s) ds$$

OR Euler approximations (Langevin diff. $\rightsquigarrow$ Metropolis-Hasting moves)
OR Fully continuous time particle models $\rightsquigarrow$ Schrödinger operators

$$\frac{d}{dt}\gamma_t(f) = \gamma_t(L_t^V(f)) \quad \text{with} \quad L_t^V = L'_t + V_t$$

$$\gamma_t(1) = \mathbb{E}\left(\exp \int_0^t V_s(X'_s) ds\right) = \exp \int_0^t \eta_s(V_s) ds \quad \text{with} \quad \eta_t = \gamma_t / \gamma_t(1)$$

$\supset$ *Continuous time models* $\supset$ *Langevin diffusions*

$$X_n := X'_{[t_n,t_{n+1}[} \quad \& \quad G_n(X_n) = \exp \int_{t_n}^{t_{n+1}} V_s(X'_s)ds$$

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$$\frac{d}{dt}\gamma_t(f) = \gamma_t(L^V_t(f)) \quad \text{with} \quad L^V_t = L'_t + V_t$$

$$\gamma_t(1) = \mathbb{E}\left(\exp \int_0^t V_s(X'_s)ds\right) = \exp \int_0^t \eta_s(V_s)ds \quad \text{with} \quad \eta_t = \gamma_t/\gamma_t(1)$$

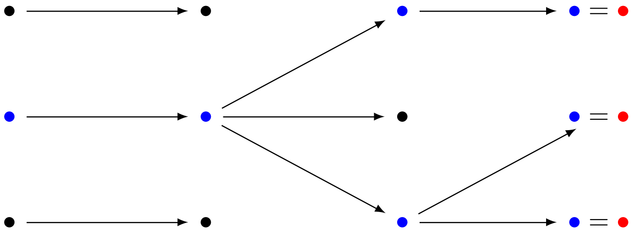
Master equation $\eta_t = \mathrm{Law}(\overline{X}_t) \Rightarrow \frac{d}{dt}\eta_t(f) = \eta_t(L_{t,\eta_t}(f))$
(ex. : $V_t = -U_t \leq 0$)

$$L_{t,\eta_t}(f)(x) = \underbrace{L'_t(f)(x)}_{\text{free exploration}} + \underbrace{U_t(x)}_{\text{acceptance rate}} \int (f(y) - f(x)) \underbrace{\eta_t(dy)}_{\text{interacting jump law}}$$

$$\Downarrow$$

Particle model: Survival-acceptance rates $\oplus$ Recycling jumps

Genealogical tree evolution $(N, n) = (3, 3)$

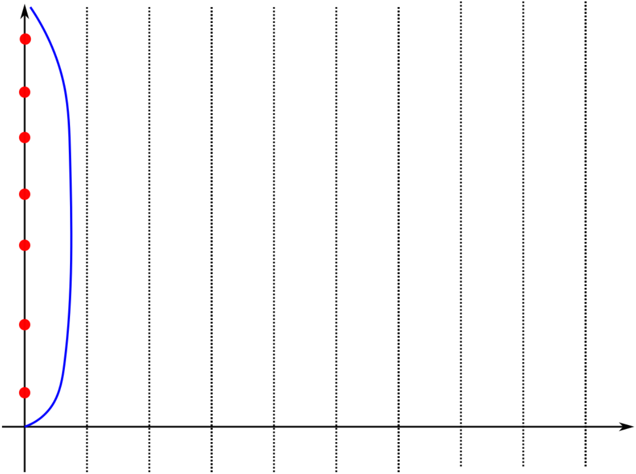


Some particle estimates $(\delta_a(dx) \leftrightarrow \delta(x-a)\ dx)$

- Individuals ξ_n^i "almost" iid with law $\eta_n \simeq \eta_n^N = \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$
- Ancestral lines "almost" iid with law $\mathbb{Q}_n \simeq \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\mathrm{line}_n(i)}$
- Normalizing constants

$$\mathcal{Z}_{n+1} = \prod_{0\leq p\leq n}\eta_p(G_p) \simeq_{N\uparrow\infty} \mathcal{Z}_{n+1}^N = \prod_{0\leq p\leq n}\eta_p^N(G_p) \quad (\text{Unbiased})$$

Graphical illustration : $\eta_n \simeq \eta_n^N := \frac{1}{N}\sum_{1\leq i\leq N}\delta_{\xi_n^i}$



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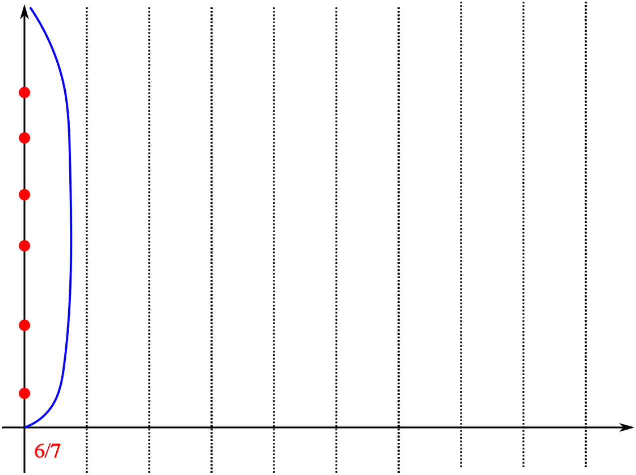

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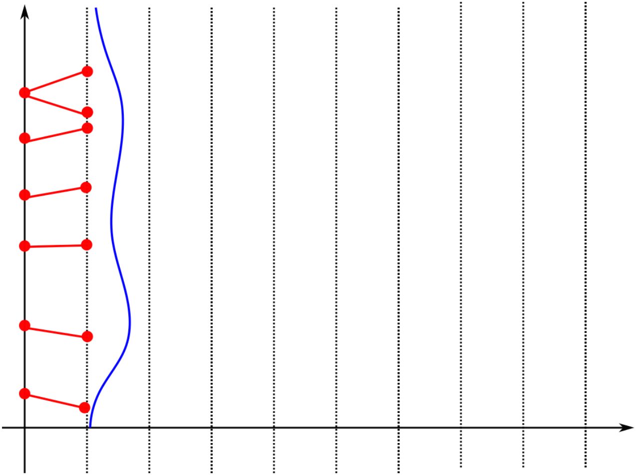

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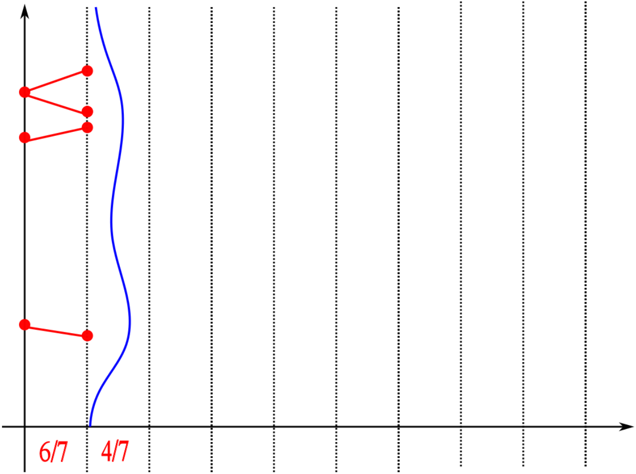

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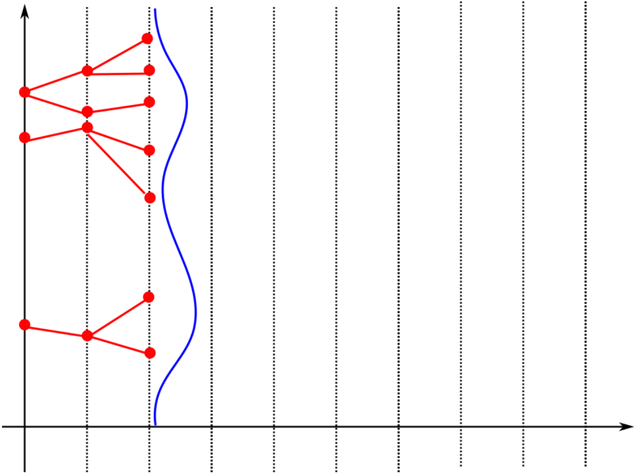

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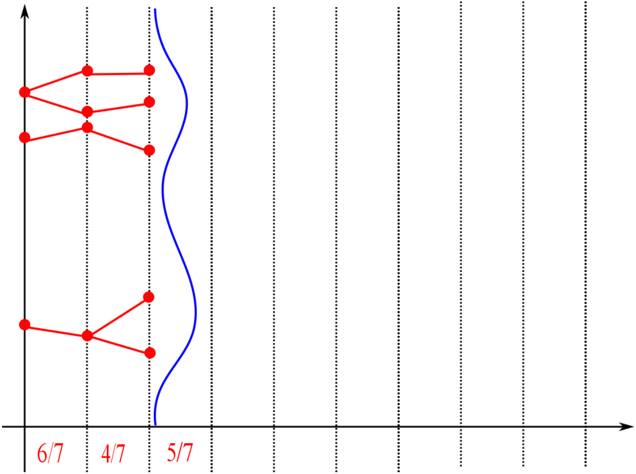

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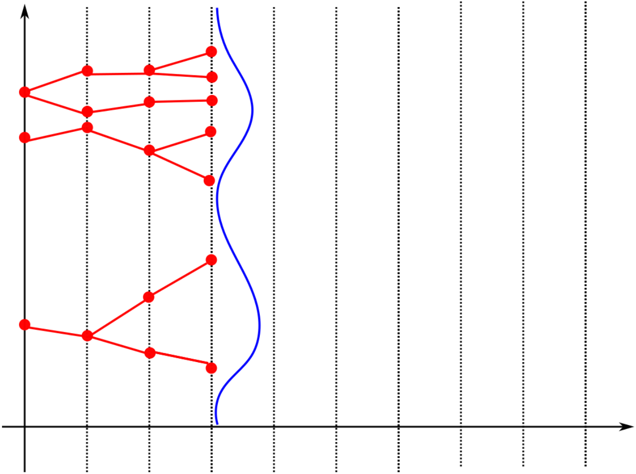

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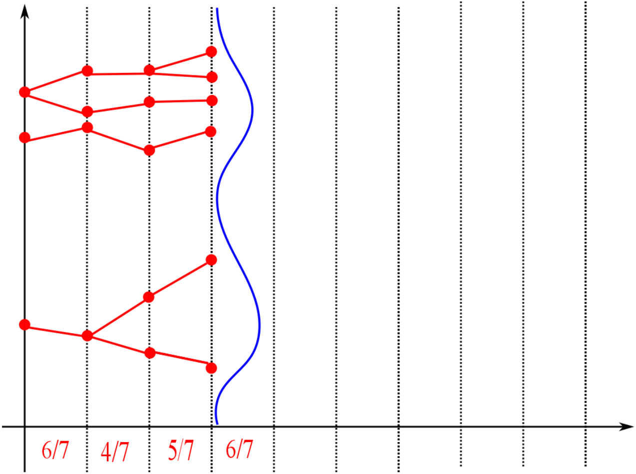

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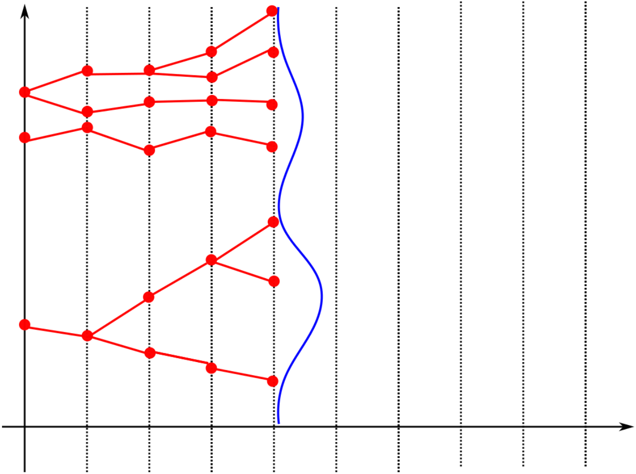

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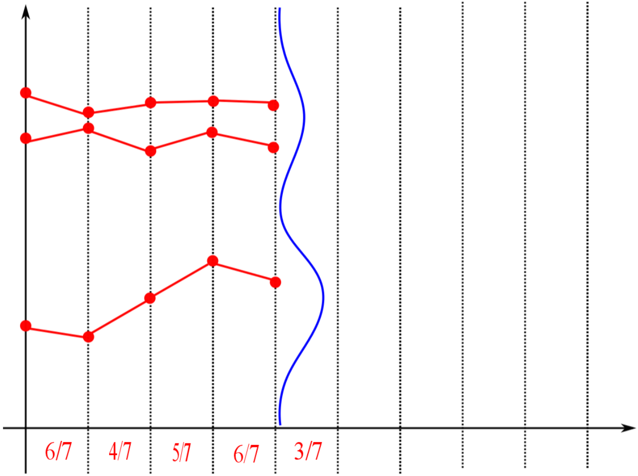

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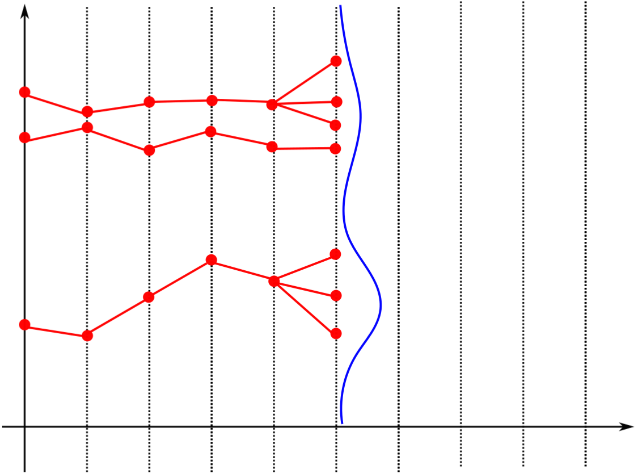

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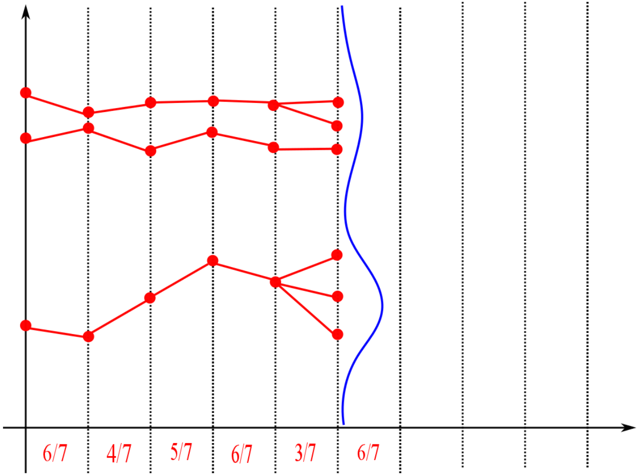

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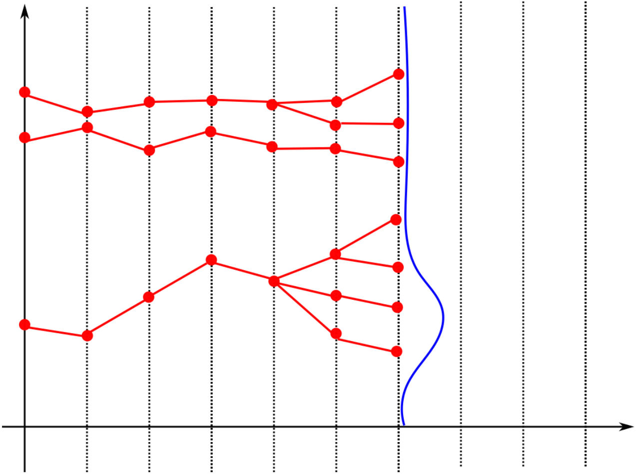

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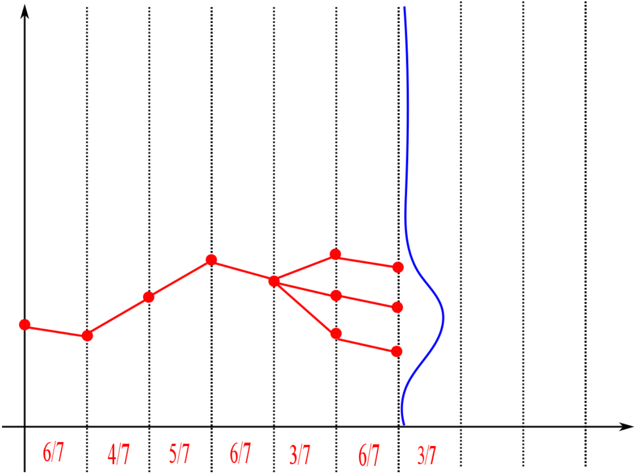

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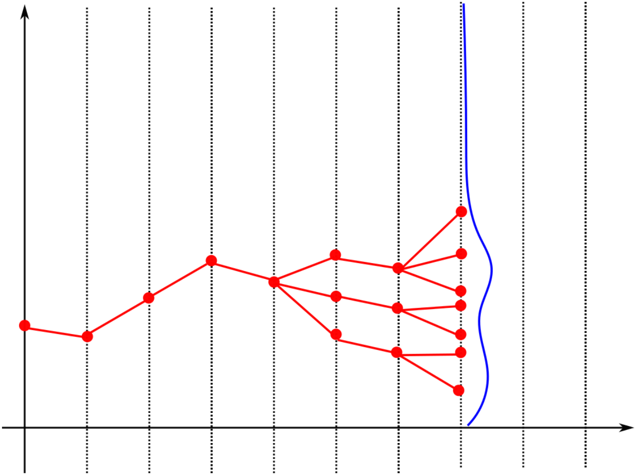

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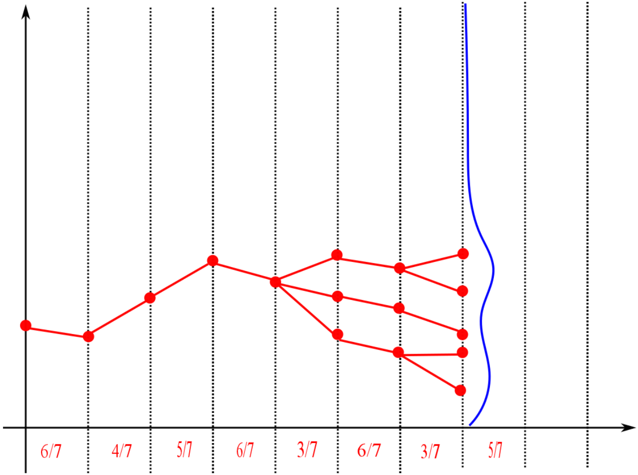

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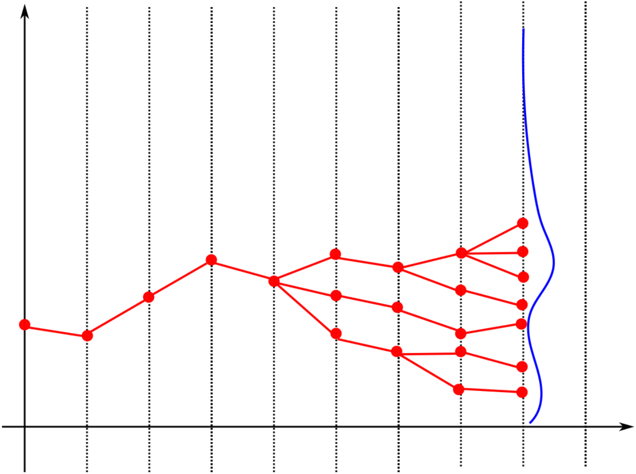

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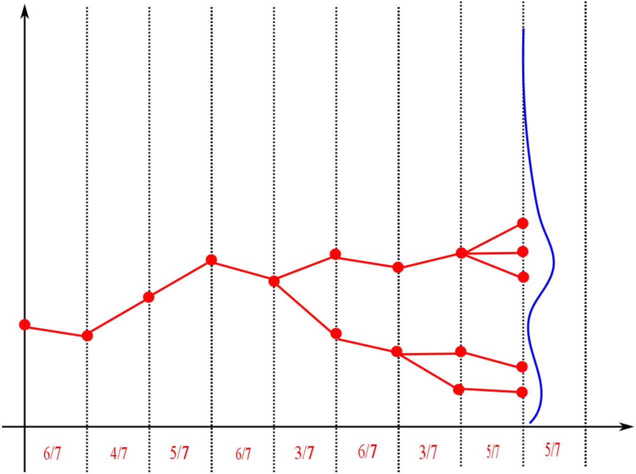

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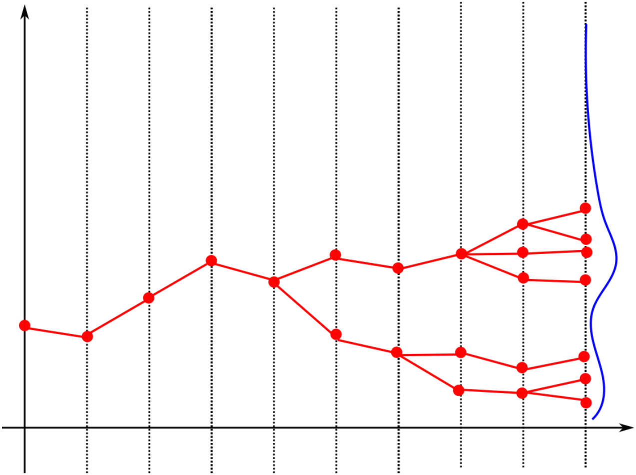

How to use the full ancestral tree model ?

$$G_{n-1}(x_{n-1})M_n(x_{n-1}, dx_n) \stackrel{hyp}{=} H_n(x_{n-1}, x_n)\ \nu_n(dx_n)$$

$\Rightarrow$ **Backward Markov model :**

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) = \eta_n(dx_n) \quad \underbrace{\mathbb{M}_{n,\eta_{n-1}}(x_n, dx_{n-1})}_{\propto\ \eta_{n-1}(dx_{n-1})\ H_n(x_{n-1}, x_n)} \ldots \mathbb{M}_{1,\eta_0}(x_1, dx_0)$$

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Particle approximation

$$\mathbb{Q}_n^N(d(x_0,\ldots,x_n)) = \eta_n^N(dx_n)\ \mathbb{M}_{n,\eta_{n-1}^N}(x_n,dx_{n-1})\ldots\mathbb{M}_{1,\eta_0^N}(x_1,dx_0)$$

Ex.: Additive functionals $\quad \mathbf{f_n}(x_0,\ldots,x_n) = \frac{1}{n+1}\sum_{0\le p\le n} f_p(x_p)$

$$\mathbb{Q}_n^N(\mathbf{f_n}) := \frac{1}{n+1}\sum_{0\le p\le n}\eta_n^N\underbrace{\mathbb{M}_{n,\eta_{n-1}^N}\ldots\mathbb{M}_{p+1,\eta_p^N}(f_p)}_{\text{matrix operations}}$$

Self avoiding walks in $\mathbb{Z}^d$

Feynman-Kac model with

$$\mathbf{X_n} = (X_0, \ldots, X_n) \quad \& \quad G_n(\mathbf{X_n}) = 1_{X_n \not\in \{X_0, \ldots, X_{n-1}\}}$$

$$\Updownarrow$$

Conditional distributions

$$\mathbb{Q}_n = \mathrm{Law}\left((\mathbf{X_0}, \ldots, \mathbf{X_n}) \mid X_p \neq X_q, \ \forall 0 \leq p < q < n\right)$$

and

$$\mathcal{Z}_n = \mathrm{Proba}\left(X_p \neq X_q, \ \forall 0 \leq p < q < n\right)$$

Level crossing probabilities (1)

$$\mathbb{P}\left(V_n(X_n) \geq a\right) \quad \text{or} \quad \mathbb{P}\left(X \text{ hits } A_n \text{ before } B\right)$$

- Level crossing at a fixed given time

$$\begin{aligned}\mathbb{P}\left(V_n(X_n) \geq a\right) &= \mathbb{E}\left(f_n(X_n)\; e^{V_n(X_n)}\right) \\ &= \mathbb{E}\left(\mathbf{f_n}(\mathbf{X}_n)\; \prod_{0\leq p<n} G_p(\mathbf{X}_p)\right)\end{aligned}$$

with

- The Markov chain on transition space

$$\mathbf{X}_n = (X_n, X_{n+1}) \quad \text{and} \quad G_n(\mathbf{X}_n) = \exp\left[V_{n+1}(X_{n+1}) - V_n(X_n)\right]$$

- The test functions

$$f_n(\mathbf{X}_n) = 1_{V_n(X_n)\geq a}\; e^{-V_n(X_n)}$$

Level crossing probabilities (2)

- Excursion level crossing $A_n \downarrow$, with B non critical recurrent subset.

$$\mathbb{P}(X \text{ hits } A_n \text{ before } B) \;=\; \mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right)$$

$$T_n := \inf\{p \geq T_{n-1} \;:\; X_p \in (A_n \cup B)\}$$

Feynman-Kac model

$$\mathbb{E}\left(\prod_{0\leq p\leq n} 1_{A_p}(X_{T_p})\right) = \mathbb{E}\left(\prod_{0\leq p<n} G_p(\mathbf{X}_p)\right)$$

with

$$\mathbf{X}_n = (X_p)_{p\in[T_n, T_{n+1}]} \quad \& \quad G_n(\mathbf{X}_n) = 1_{A_{n+1}}(X_{T_{n+1}})$$

Absorption models

- Sub-Markov semigroups

$$Q_n(x, dy) = G_{n-1}(x)\ M_n(x, dy) \rightsquigarrow E_n^c = E_n \cup \{c\}$$

- Absorbed Markov chain

$$X_n^c \in E_n^c \xrightarrow{\text{absorption } \sim (1-G_n)} \widehat{X}_n^c \xrightarrow{\text{exploration } \sim M_n} X_{n+1}^c$$

$$\Downarrow$$

$$\mathbb{Q}_n = \mathrm{Law}((X_0^c, \ldots, X_n^c) \mid T^{absorption} \geq n)$$

and

$$\mathcal{Z}_n = \mathrm{Proba}\left(T^{absorption} \geq n\right)$$

Homogeneous models $(G_n, M_n) = (G, M)$

- Reversibility condition : $\mu(dx)M(x,dy) = \mu(dy)M(y,dx)$

$$\mathrm{Proba}\left(T^{absorption} \geq n\right) \simeq \lambda^n$$

with $\lambda =$ top eigenvalue of

$$Q(x,dy) = G(x)\ M(x,dy)$$

- $Q(h) = \lambda h$
  - Quasi-invariant measure :

$$\mathbb{P}(X_n^c \in dx \mid T^{absorption} > n) \to_{n\uparrow} \frac{1}{\mu(h)}\ h(x)\ \mu(dx)$$

  - Doob h-process X^h :

$$M^h(x,dy) = \frac{1}{\lambda} h^{-1}(x) Q(x,dy) h(y) = \frac{Q(x,dy)h(y)}{Q(h)(x)} = \frac{M(x,dy)h(y)}{M(h)(x)}$$

Homogeneous models $(G_n, M_n) = (G, M)$

$$\mathbb{Q}_n(d(x_0, \ldots, x_n)) \propto \mathbb{P}((X_0^h, \ldots, X_n^h) \in d(x_0, \ldots, x_n))\ h^{-1}(x_n)$$

- *Invariant measure $\mu_h = \mu_h M^h$ & normalized additive functionals*

$$\overline{F}_n(x_0, \ldots, x_n) = \frac{1}{n+1} \sum_{0 \le p \le n} f(x_p) \Longrightarrow \mathbb{Q}_n(\overline{F}_n) \simeq_n \mu_h(f)$$

- If $G = G^\theta$ depends on some $\theta \in \mathbb{R} \rightsquigarrow f := \frac{\partial}{\partial \theta} \log G^\theta$

$$\frac{\partial}{\partial \theta} \log \lambda^\theta \simeq_n \frac{1}{n+1} \frac{\partial}{\partial \theta} \log \mathcal{Z}_{n+1}^\theta = \mathbb{Q}_n(\overline{F}_n)$$

NB : Similar expression when M^θ depends on some $\theta \in \mathbb{R}$.

Semigroup gradient estimates

$$X_{n+1}(x) = \mathcal{F}_n(X_n(x), W_n) \quad \left(X_0(x) = x \in \mathbb{R}^d\right) \quad \rightsquigarrow \quad P_n(f)(x) := \mathbb{E}\left(f(X_n(x))\right)$$

First variational equation

$$\frac{\partial X_{n+1}}{\partial x}(x) = A_n(x, W_n)\ \frac{\partial X_n}{\partial x}(x) \quad \text{with} \quad A_n^{(i,j)}(x,w) = \frac{\partial \mathcal{F}_n^i(\,.\,, w)}{\partial x^j}(x)$$

Random process on the sphere $U_0 = u_0 \in \mathbb{S}^{d-1}$

$$U_{n+1} = A_n(X_n, W_n)U_n / \|A_n(X_n, W_n)U_n\| = \frac{\frac{\partial X_n}{\partial x}(x)\ u_0}{\left\|\frac{\partial X_n}{\partial x}(x)\ u_0\right\|}$$

Feynman-Kac model $\mathcal{X}_n = (X_n, U_n, W_n)$ **&** $\mathcal{G}_n(x,u,w) = \|\mathcal{A}_n(x,w)\ u\|$

$$\nabla P_{n+1}(f)(x)\ u_0 = \mathbb{E}\left(\underbrace{F(\mathcal{X}_{n+1})}_{\nabla f(X_{n+1})\ U_{n+1}} \quad \underbrace{\prod_{0 \le p \le n} \mathcal{G}_p(\mathcal{X}_p)}_{\left\|\frac{\partial X_n}{\partial x}(x)\ u_0\right\|} \right)$$

Boltzmann-Gibbs measures

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n}\, e^{-\beta_n V(x)}\, \lambda(dx) \quad \text{with} \quad \beta_n \uparrow$$

- For any MCMC transition M_n with target η_n

$$\eta_n = \eta_n M_n$$

- Updating of the temperature parameter

$$\eta_{n+1} = \Psi_{G_n}(\eta_n) \quad \text{with} \quad G_n = e^{-(\beta_{n+1}-\beta_n)V}$$

Proof : $\quad e^{-\beta_{n+1}V} = e^{-(\beta_{n+1}-\beta_n)V} \times e^{-\beta_n V}$

Consequence :

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

and $(\beta_0 = 0)$

$$\lambda\left(e^{-\beta_n V}\right) = \mathcal{Z}_n = \prod_{0 \le p < n} \eta_p(G_p)$$

Restriction models

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n}\, 1_{A_n}(x)\ \lambda(dx) \quad \text{with} \quad A_n \downarrow$$

- For any MCMC transition M_n with target η_n

$$\eta_n = \eta_n M_n$$

- Updating of the subset

$$\eta_{n+1} = \Psi_{G_n}(\eta_n) \quad \text{with} \quad G_n = 1_{A_{n+1}}$$

Proof : $\quad 1_{A_{n+1}} = 1_{A_{n+1}} \times 1_{A_n}$

Consequence :

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

and $(\lambda(A_0) = 1)$

$$\lambda(A_n) = \mathcal{Z}_n = \prod_{0 \le p < n} \eta_p(G_p)$$

Product models

$$\eta_n(dx) := \frac{1}{\mathcal{Z}_n}\left\{\prod_{p=0}^{n} h_p(x)\right\}\lambda(dx) \quad \text{with} \quad h_p \geq 0$$

- For any MCMC transition M_n with target $\eta_n = \eta_n M_n$.
- Updating of the product

$$\eta_{n+1} = \Psi_{G_n}(\eta_n) \quad \text{with} \quad G_n = h_{n+1}$$

Proof : $\left\{\prod_{p=0}^{n+1} h_p\right\} = h_{n+1} \times \left\{\prod_{p=0}^{n} h_p\right\}$

Consequence :

$$\eta_{n+1} = \eta_{n+1} M_{n+1} = \Psi_{G_n}(\eta_n) M_{n+1}$$

and $(h_0 = 1)$

$$\lambda\left(\prod_{p=0}^{n} h_p\right) = \mathcal{Z}_n = \prod_{0 \leq p < n} \eta_p(G_p)$$

Equivalent particle algorithms

Sequential Monte Carlo	Sampling	Resampling
Particle Filters	Prediction	Updating
Genetic Algorithms	Mutation	Selection
Evolutionary Population	Exploration	Branching
Diffusion Monte Carlo	Free evolutions	Absorption
Quantum Monte Carlo	Walkers motions	Reconfiguration
Sampling Algorithms	Transition proposals	Accept-reject-recycle

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More botanical names:
bootstrapping, spawning, cloning, pruning, replenish, multi-level splitting, enrichment, go with the winner, ...

$1950 \leq$ Heuristic style algo. $\leq 1996 \leq$ Particle Feynman-Kac models

Convergence analysis : CLT, LDP, $\mathbb{L}_p$-estimates, Empirical processes, Moderate deviations, propagations of chaos, exact weak expansions, ...

Concentration analysis = Exponential deviation proba. estimates

Stochastic linearization/Mean field particle models

- Discrete time models ($\eta_n = \mathrm{Law}(\overline{X}_n)$)

$$\begin{aligned}\eta_n &= \eta_{n-1} K_{n,\eta_{n-1}} \rightsquigarrow \text{transition} \quad \xi^i_n \sim K_{n,\eta^N_{n-1}}\left(\xi^i_{n-1}, dx_n\right)\\ \eta^N_n &= \eta^N_{n-1} K_{n,\eta^N_{n-1}} + \frac{1}{\sqrt{N}}\, W^N_n\end{aligned}$$

Theo : $\left(W^N_n\right)_{n\geq 0} \rightarrow (W_n)_{n\geq 0} \perp$ centered Gaussian fields

Stochastic linearization/Mean field particle models

- Discrete time models ($\eta_n = \mathrm{Law}(\overline{X}_n)$)

$$\begin{aligned}
\eta_n &= \eta_{n-1} K_{n,\eta_{n-1}} \rightsquigarrow \text{transition} \quad \xi_n^i \sim K_{n,\eta_{n-1}^N}\left(\xi_{n-1}^i, dx_n\right) \\
\eta_n^N &= \eta_{n-1}^N K_{n,\eta_{n-1}^N} + \frac{1}{\sqrt{N}}\; W_n^N
\end{aligned}$$

Theo : $\left(W_n^N\right)_{n\geq 0} \to (W_n)_{n\geq 0} \perp$ centered Gaussian fields

- Continuous time models ($\eta_t = \mathrm{Law}(\overline{X}_t)$)

$$\begin{aligned}
\frac{d}{dt}\eta_t(f) &= \eta_t(L_{t,\eta_t}(f)) \rightsquigarrow \text{generator} \quad \xi_t^i \sim L_{t,\eta_t^N} \\
d\eta_t^N(f) &= \eta_t^N(L_{t,\eta_t^N}(f))\; dt + \frac{1}{\sqrt{N}}\; dM_t^N(f)
\end{aligned}$$

Theo : $M_t^N(f) \to M_t$ Gaussian martingale with

$$d\langle M(f)\rangle_t = \eta_t(\Gamma_{L_{t,\eta_t}}(f,f))\; dt$$

Current population models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time.
Test funct. $\|f_n\| \leq 1$, $\forall\ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \eta_n\right](f) \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $x = (x_i)_{1 \leq i \leq d} \rightsquigarrow (-\infty, x] = \prod_{i=1}^{d}(-\infty, x_i]$ cells in $E_n = \mathbb{R}^d$.

$$F_n(x) = \eta_n\left(1_{(-\infty,x]}\right) \quad \text{and} \quad F_n^N(x) = \eta_n^N\left(1_{(-\infty,x]}\right)$$

  The probability of the following event

$$\sqrt{N}\ \left\|F_n^N - F_n\right\| \leq c\ \sqrt{d\ (x+1)}$$

  is greater than $1 - e^{-x}$.

Particle free energy models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time
$\forall\ (x \geq 0, n \geq 0, N \geq 1)$

- Unbiased property

$$\mathbb{E}\left(\eta_n^N(f_n) \prod_{0 \leq p < n} \eta_p^N(G_p)\right) = \mathbb{E}\left(f_n(X_n) \prod_{0 \leq p < n} G_p(X_p)\right)$$

- For any $\epsilon \in \{+1, -1\}$, the probability of the event

$$\frac{\epsilon}{n} \log \frac{\mathcal{Z}_n^N}{\mathcal{Z}_n} \leq \frac{c_1}{N}\ \left(1 + x + \sqrt{x}\right) + \frac{c_2}{\sqrt{N}}\ \sqrt{x}$$

is greater than $1 - e^{-x}$.

note $(0 \leq \epsilon \leq 1 \Rightarrow (1 - e^{-\epsilon}) \vee (e^{\epsilon} - 1) \leq 2\epsilon)$

$$e^{-\epsilon} \leq \frac{z^N}{z} \leq e^{\epsilon} \Rightarrow \left|\frac{z^N}{z} - 1\right| \leq 2\epsilon$$

Genealogical tree models $:= \eta_n^N$ (in path space)

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time
$\mathbf{f_n}$ test function $\|\mathbf{f_n}\| \leq 1$, $\forall \ (x \geq 0, n \geq 0, N \geq 1)$.

- The probability of the event

$$\left[\eta_n^N - \mathbb{Q}_n\right](f) \leq c_1 \ \frac{n+1}{N} \ \left(1 + x + \sqrt{x}\right) + c_2 \ \sqrt{\frac{(n+1)}{N}} \ \sqrt{x}$$

  is greater than $1 - e^{-x}$.

- $\mathcal{F}_n =$ indicator fct. $\mathbf{f_n}$ of cells in $\mathbf{E_n} = \left(\mathbb{R}^{d_0} \times \ldots, \times \mathbb{R}^{d_n}\right)$
  The probability of the following event

$$\sup_{\mathbf{f_n} \in \mathcal{F}_n} \left|\eta_n^N(\mathbf{f_n}) - \mathbb{Q}_n(\mathbf{f_n})\right| \leq c \ (n+1) \ \sqrt{\frac{\sum_{0 \leq p \leq n} d_p}{N} \ (x+1)}$$

  is greater than $1 - e^{-x}$.

Backward particle models

Constants (c_1, c_2) related to (bias,variance), c universal constant $\perp$ time.
$\mathbf{f_n}$ normalized additive functional with $\|f_p\| \leq 1$, $\forall$ $(x \geq 0, n \geq 0, N \geq 1)$

- The probability of the event

$$\left[\mathbb{Q}_n^N - \mathbb{Q}_n\right](\overline{\mathbf{f}}_n) \leq c_1 \, \frac{1}{N} \, (1 + (x + \sqrt{x})) + c_2 \, \sqrt{\frac{x}{N(n+1)}}$$

  is greater than $1 - e^{-x}$.

- $\mathbf{f}_{a,n}$ normalized additive functional w.r.t. $f_p = 1_{(-\infty,a]}$, $a \in \mathbb{R}^d = E_n$
  .
  The probability of the following event

$$\sup_{a \in \mathbb{R}^d} \left|\mathbb{Q}_n^N(\mathbf{f_{a,n}}) - \mathbb{Q}_n(\mathbf{f_{a,n}})\right| \leq c \, \sqrt{\frac{d}{N}(x+1)}$$

  is greater than $1 - e^{-x}$.