PHYSICS : PARADIGM OF CURRENT (or TRAFFIC)

DISCRETE TIME :
$$J_N^{f} = \frac{1}{N} \sum_{i=1}^{N} f(X_i, X_{i+1})$$

CONTINUOUS TIME :

⊖ Ch Markov Temps Continu

$$J_T^{f} = \frac{1}{T} \sum_{\substack{0 \le s \le T \\ \Delta X_s \neq 0}} f(X_{s^-}, X_{s^+}) + \frac{1}{T} \int_0^T g(X_s)\, ds$$

⊖ DIFFUSION PROCESS

$$J_T^{f} = \frac{1}{T} \int_0^T \left(f(X_s) \circ dX_s + g(X_s)\, ds \right)$$

EX : – ENTROPY CREATION : $f(x,y) = \ln \frac{W(x,y)}{W(y,x)}$

– CURRENT OF PARTICLES

⇒ CAN WE GO BEYOND ?

AT DISCRETE TIME : YES : m-Word empirical DENSITY.

LEVEL 1

$$O_N^e = \frac{1}{N}\sum_{i=1}^{N} O(x_i)$$

$$O_T^e = \frac{1}{T}\int_0^T O(x_s)\, ds$$

LEVEL 2

EMPIRICAL DENSITY

$$\mu_N^e = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i}$$

$\downarrow$ TYPICAL

μ_{INV}

$$\mu_T^e = \frac{1}{T}\int_0^T ds\, \delta_{x_s}$$

DONSKER-VARADHAN

$\Downarrow$

NOT EXPLICIT, EXCEPT FOR:

- ⊕ IID CASE (SANOV)
- ⊕ REVERSIBLE CONTINUOUS-TIME
- ⊕ SPECIAL CASE

DONSKER-VARADHAN PROVE OF THE LEVEL 2

LEVEL 3

EMPIRICAL PROCESS

$$\mu_N^{e,3} = \frac{1}{N}\sum_{i=1}^{N} \delta_{\theta^i(x)} \xrightarrow{\text{TYPICAL}} \mathbb{P}_{\mu_{INV}}$$

MESURE STATIONNAIRE MAIS NON MARKOV

EXPLICIT

NOT VERY INTUITIVE

Sanov theorem for independents bits : Combinatorial approach

- 2 states 0 and 1, $P(0) = p_0$ and $P(1) = p_1 = 1 - p_0$
- Sequence of n bits : $[x] = 011100100....$
- Empirical density ("type" in statistics) : $\rho_n^e(0) = \frac{\sharp 0 \text{ in } [x]}{n}$ and $\rho_n^e(1) = \frac{\sharp 1 \text{ in } [x]}{n}$
- Typical behavior : $\rho_n^e(0) \to p_0$ and $\rho_n^e(1) \to p_1$.
- Untypical behavior $P\left(\rho_n^e(0) = \mu_0, \rho_n^e(1) = \mu_1\right) = \frac{n!}{(n\mu_0)!(n\mu_1)!}$ $\mathrm{p}_0^{n\mu_0} p_1^{n\mu_1}$

with $\mu_0 + \mu_1 = 1$.

- Stirling : $\ln\left(P\left(\rho_n^e(0) = \mu_0, \rho_n^e(1) = \mu_1\right)\right) = n\mu_0 \ln p_0 - n\mu_0 + n\mu_1 \ln p_1 - n\mu_1 + n \ln n - n - n\mu_0 \ln n\mu_0 + n\mu_0 - n\mu_1 \ln n\mu_1 + n\mu_1 = -n\left(\mu_0 \ln \frac{\mu_0}{p_0} + \mu_1 \ln \frac{\mu_1}{p_1}\right)$

$$\Rightarrow I(\mu) = \mu_0 \ln \frac{\mu_0}{p_0} + \mu_1 \ln \frac{\mu_1}{p_1} \equiv S(\mu/p)$$

Methods : Not rigorous mathematics...

La physique théorique est devenue un vaste hôpital psychiatrique où ce sont les fous qui ont pris le pouvoir. La théorie des cordes ? Une physique sans expérience et une mathématique sans rigueur.
Jean Marie Souriau (1922-2012(March)) :

Positive description :

Physics is to mathematics as ♡ is to masturbation. Richard Feymann

(Totally ?) No rigorous prove of Donsker-Varadhan at the Physical way

- Functional Laplace transformation
$\left\langle \delta\left(\rho^{e}_{1,T}-\rho\right)\right\rangle = \left\langle \int dV \exp\left(-(TV,\rho^{e}_{1,T}-\rho)\right)\right\rangle =$
$\int dV \exp\left(T\int dx V(x)\rho(x)\right)\left\langle \exp\left(-\int_0^T dt V(X_t)\right)\right\rangle$
- Feymann-Kac formulae :
$\left\langle \exp\left(-\int_0^T dt V(X_t)\right)\right\rangle = \int \rho_0(dx)dy \exp\left(T(L-V)\right)(x,y) =$
$\int \rho_0(dx)dy \exp\left(-T(-L+V)\right)(x,y) \approx \exp\left(-T\lambda_+[V]\right)$ where we define
$\lambda_+[V] = \inf\left[Spectre\left(-L+V\right)\right]$
- Saddle point :
$\left\langle \delta\left(\rho^{e}_{1,T}-\rho\right)\right\rangle \asymp \exp\left(T\inf_{[V]\in R}\left(\int dx V(x)\rho(x) - \lambda_+[V]\right)\right)$ then
$I[\rho] = -\inf_{[V]\in R}\left(\int dx\rho(x)\left(V(x)-\lambda_+[V]\right)\right)$.
- $\rho_+[V]$ the eigenvalue associated $\lambda_+[V]$, by Perron Frobenius theorem
$\rho_+[V](x) \geq 0$ for all x and $\lambda_+[V] \geq 0.$ $\Rightarrow$
$(-L+V)\rho_+[V] = \lambda_+[V]\rho_+[V]$ then $V-\lambda_+[V] = \left(\rho_+[V]\right)^{-1}L\rho_+[V]$
- Then we obtain the rate function

$$\boldsymbol{I[\rho] = -\inf_{[u]\in R^+}\left(\int dx\rho(x)\left(u^{-1}Lu\right)(x)\right) = -\inf_{[f]\in R}\left(\int dx\rho(x)\left(\exp(-f)L\exp f\right)(x)\right)}$$

LEVEL 2,5

LEVEL OF THE CURRENTS

$$\mu_N^{e,2} = \frac{1}{N}\sum_{i=1}^{N} \delta_{x_i, x_{i+1}}$$

$\downarrow$

$\nu_{Inv} \otimes M$

CTMC: $\rho_T^{e}(x) = \frac{1}{T}\int_0^T \delta_{x_s}(x)$

$$\rho_T^{e,2}(x,y) = \frac{1}{T}\sum_{\substack{0<s<T \\ \Delta x_s \neq 0}} \delta_{x_{s^-}}(x)\,\delta_{x_{s^+}}(y)$$

$$I[\mu] = \sum_{x,y} \mu(x,y) \ln\left[\frac{\mu(x,y)}{\sum_z \mu(x,z)\, M(x,y)}\right]$$

METHOD OF PROVE

1. CONTRACTION LEVEL 3 (WHERE?) → HARD (NOT CONTINUOUS)
2. LEVEL 2 FOR $Y_n = (X_n, X_{n+1})$ (DZ) → NO
3. GARTNER-ELLIS ⊕ FEYNMANN-KAC (DEN HOLLANDER) → YES (WHERE?)
4. TILTING
   - IID (DEN HOLLANDER) → NO
   - PROCESS FOR WHICH μ IS TYPICAL → YES (MES)

LEVEL 2,999...9...

m-words

1. YES
2. YES
3. NO
4. ↗ YES ↘ NO !!!!

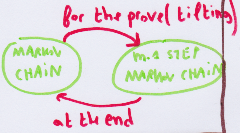
1. NO
2. NO
3. NO
4. NO NO

??

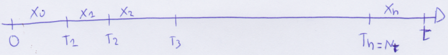
LEVEL 2,9999.....

NEW METHOD:

(5) m-1 STEP MARKOV CHAIN



$\Longrightarrow$ YES



$$\sigma^{e,m-1}_T \equiv \frac{1}{T}\sum_{i=0}^{N_T-m+2} \delta_{x_i,\dots,x_{i+m-2}}\left(T_{i+m-1}-T_{i+m-2}\right)$$

$$\rho^{e,m}_T = \frac{1}{T}\sum_{i=0}^{N_T-m+1} \delta_{x_i,\dots,x_{i+m-1}}$$

$$I[\sigma,\rho] \equiv \sum_{x_1,\dots,x_m}\left[\sigma(x_1,\dots,x_{m-1})W(x_{m-1},x_m) - \rho(x_1,\dots,x_m) + \rho(x_1,\dots,x_m)\ln\left(\frac{\rho(x_1,\dots,x_m)}{\sigma(x_1,\dots,x_{m-1})W(x_{m-1},x_m)}\right)\right]$$

Level 2.99999.... : Context

- X_t Continuous time Markov chain with transition rate $W(x,y)$ and escape rate $\lambda(x) = \sum_y W(x,y)$ on a state space V.
- ρ_{inv} its invariant density.
- $\mathbb{P}_x$ the law of the process with initial condition x, which is a distribution in the Skorohod space $D(\mathbb{R}_+, V)$
- X_n Included Markov chain.
  Markov matrix : $M(x,y) = \frac{W(x,y)}{\lambda(x)}$.
  Invariant density π_{inv} and it is easy to see that
  $$\pi_{inv}(x) = \frac{\rho_{inv}(x)\lambda(x)}{\langle\lambda\rangle_{\rho_{inv}}}$$
- We note the sequence of time of jump of the process $T_1, T_2, ..., T_n, ...$

Level 2.99999.... : Empirical density

$$\rho_t^{e,m} \quad \equiv \quad \frac{1}{t} \sum_{i=0}^{N_t - m+1} \delta_{X_i, X_{i+1}, \ldots, X_{i+m-1}}$$

and

$$\sigma_t^{e,m} \quad \equiv \quad \frac{1}{t} \sum_{i=0}^{N_t - m+1} \delta_{X_i, X_{i+1}, \ldots, X_{i+m-1}} \left(T_{m+i} - T_{m+i-1} \right)$$

By exemple,

$$\rho_t^{e,2} \quad \equiv \quad \frac{1}{t} \sum_{i=0}^{N_t - 1} \delta_{X_i, X_{i+1}}$$

is the "empirical jump", and

$$\sigma_t^{e,1} \quad \equiv \quad \frac{1}{t} \sum_{i=0}^{N_t} \left(T_{1+i} - T_i \right) \delta_{X_j}$$

is the empirical density.

Main result

The family of probability measure $\left\{\mathbb{P}_x \circ \left(\sigma_t^{e,m-1}, \rho_t^{e,m}\right)^{-1}\right\}_{t>0}$ satisfy as $t \to \infty$ a large deviation principle, uniformly with respect to x, with rate function :

$$I\left[\sigma, \rho\right] = \sum_{x_1,\dots,x_m} \sigma(x_1, ..., x_{m-1}) W(x_{m-1}, x_m) - \rho(x_1, ..., x_{m-1}, x_m) + \rho(x_1, ..., x_m) \ln\left[\frac{\rho(x_1, ...x_m)}{\sigma(x_1, ..., x_{m-1}) W(x_{m-1}, x_m)}\right]$$

if

$$\sum_x \rho(x, x_2, ...x_m) = \sum_x \rho(x_2, x_3, ..., x_m, x)$$

Typical behavior of the empirical density

- $\rho_t^{e,m}(x_1, ..., x_m) = \frac{N_t - m}{t} \frac{1}{N_t - m} \sum_{i=0}^{N_t - m+1} \delta_{X_i, ..., X_{i+m-1}}(x_1, ..., x_m)$
- $\frac{N_t}{t} = \sum_{x,y} \rho_t^{e,2}(x, y)$ and then $\frac{N_t}{t} \to \sum_{x,y} \rho_{inv}(x) W(x, y) = \langle \lambda \rangle_{\rho_{inv}}$
- Typical behavior of $\rho_t^{e,m}$:
  $\rho_t^{e,m}(x_1, ..., x_m) \to \langle \lambda \rangle_{\rho_{inv}} \pi_{inv}(x_1) M(x_1, x_2) M(x_2, x_3) ... M(x_{m-1}, x_m)$
  which can be also written with the transition rate

$$\rho_t^{e,m}(x_1, x_2, ..., x_m) \to \rho_{inv}(x_1) W(x_1, x_2) \frac{W(x_2, x_3)}{\lambda(x_2)} ... \frac{W(x_{m-1}, x_m)}{\lambda(x_{m-1})}.$$

- Typical behavior of $\sigma_t^{e,m}$ is

$$\sigma_t^{e,m}(x_1, x_2, ..., x_m) \quad \to \quad \rho_{inv}(x_1) \frac{W(x_1, x_2)}{\lambda(x_2)} \frac{W(x_2, x_3)}{\lambda(x_3)} ... \frac{W(x_{m-1}, x_m)}{\lambda(x_m)}$$

NOT AT ALL GENERIC FORM : TILTING IMPOSSIBLE TO DO IN THE CLASS OF MARKOV PROCESS.

$m-1-$step pure jump process

Such a process is given by 3 object :

- A $m-1$ point initial density $\pi_0(x_0, x_1, ..., x_{m-2})$
- A transition “matrix” of the included $m-1$ step markov chain $M(x_0, x_1, ..., x_{m-2} \to y)$
- A escape rate function of the state $x_0, x_1, ..., x_{m-2}$: $\lambda(x_0, x_1, ..., x_{m-2})$

”Transition rate” of the process are
$W(x_0, x_1, ..., x_{m-2}, y) = \lambda(x_0, x_1, ..., x_{m-2})M(x_0, x_1, ..., x_{m-2} \to y)$
Trajectory with n jump, which pass toward the point $x_0, x_1, ..., x_n$ and which jump at sequence of time $t_1, t_2, ..., t_n$, the law of the process on the interval $[0, T]$,
$dP_T = \pi_0(x_0, x_1, x_{m-2})M(x_0, ..., x_{m-2} \to x_{m-1})M(x_1, ..., x_{m-1} \to x_m)$
$...\mathrm{M}(x_{n-m+1}, x_{n-m+2}, ..., x_{n-1} \to x_n)$
$\lambda(x_0, x_1, x_{m-2}) \exp\left(-\lambda(x_0, x_1, x_{m-2})t_1\right) dt_1$
$\lambda(x_1, x_2, ..., x_{m-1}) \exp\left(-\lambda(x_1, x_2, ..., x_{m-1})\left(t_2 - t_1\right)\right) dt_2 ...$
$\exp\left(-\lambda(x_{n-m+1}, x_{n-m+2}, ..., x_{n-1})\left(T - t_n\right)\right) dt_n.$

TILTING POSSIBLE TO DO IN THIS CLASS

Open issue : GC in disordered system

Large deviations for a random walk in a random environment, Ann. Prob. 22 (1994),

A.Greven, F.Den Hollander.

- Random environment w_k, $k \in Z$ iid $\in [0,1]$ random variable with distribution α
- Discrete random walk X_n in Z with
  $P_w(X_{n+1} = x \pm 1 | X_n = x) = \begin{cases} w_n \\ 1 - w_n \end{cases}$
- Large deviation for the speed of the w-conditioned random walk :

$$P_w\left(\frac{X_n}{n} = v\right) \asymp \exp(-nI(v))$$

  - $I(v)$ is deterministic
  - $I(v)$ verify the "GC" symmetry

$$I(v) = I(-v) + v\left\langle \ln\left(\frac{1-w}{w}\right)\right\rangle$$