Non-classical large deviations in the AB model

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Workshop on Computation of Transition Trajectories
and Rare Events in Non-Equilibrium Systems
ENS Lyon, France
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Outline

Study

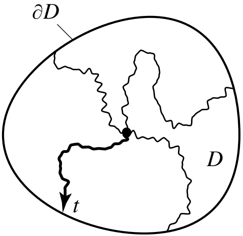
- Low-noise large deviations for stationary distribution
- Fluctuation paths – instantons
- Nonequilibrium case
- Non-isolated attractor

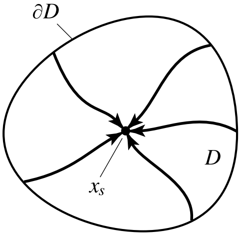
Plan

1. Recap on Freidlin-Wentzell theory
2. AB model – results
3. Conclusion

Freddy Bouchet (ENS Lyon), HT
Non-classical large deviations for a noisy system
with non-isolated attractors, J. Stat. Mech. P05028, 2012

Noise-perturbed dynamical systems





- Noisy system:

$$\dot{x}(t) = f(x(t)) + \sqrt{\nu}\,\xi(t)$$

- Gaussian white noise: $\xi(t)$

- Zero-noise system:

$$\dot{x}(t) = f(x(t))$$

- Fixed points: $f(x^*) = 0$
- Attractor: x_s

Interesting probabilities

- Propagator: $P(x, t | x_s, 0) \sim e^{-V(x,t)/\nu}$
- Stationary distribution: $P(x) \sim e^{-V(x)/\nu}$

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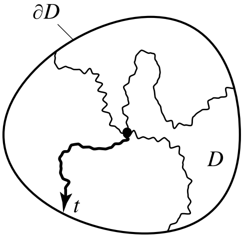

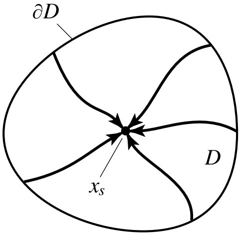

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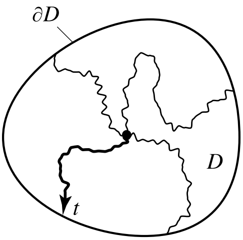
Stationary distribution

- Path integral:

$$P(x,t|x_s,0) = \int_{x_s,0}^{x,t} \mathcal{D}[x]\, P[x]$$



- Path probability:

$$P[x] \sim e^{-I[x]/\nu}, \qquad I[x] = \frac{1}{2}\int_0^t (\dot{x} - f(x))^2\, ds$$

Large deviation approximation

$$P(x) \sim e^{-V(x)/\nu}, \qquad V(x) = \inf_{x(0)=x_s, x(\infty)=x} I[x]$$

- Most probable path = min action path = instanton
- Onsager-Machlup 1950s; Graham 1980s; Freidlin-Wentzell 1970-80s
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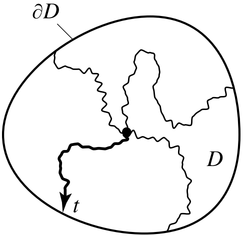

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Example: Gradient dynamics

- Gradient system:

$$\dot{x}(t) = -\nabla U(x(t)) + \sqrt{\nu}\,\xi(t)$$

- Stationary distribution:

$$P(x) \sim e^{-V(x)/\nu}, \qquad V(x) = 2U(x)$$

- Instanton = time-reverse of decay path from x to x_s
- Consequence of detailed balance
- Equilibrium system

This talk

$$P(x) \sim e^{-V(x)/\sqrt{\nu}}$$

- Non-gradient system
- Nonequilibrium system
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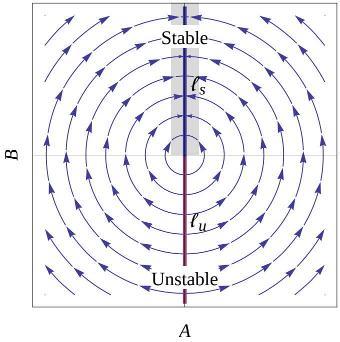
AB model

Noiseless dynamics

$$\dot{A} = -AB$$
$$\dot{B} = A^2$$

- Stable line: $A = 0, B > 0$
- Unstable line: $A = 0, B < 0$
- Energy:

$$E = A^2 + B^2, \qquad \dot{E} = 0$$



Perturbed dynamics

$$\dot{A} = -AB - \nu A + \sigma_A \sqrt{\nu}\, \xi_A(t)$$
$$\dot{B} = A^2 \quad - \nu B + \sigma_B \sqrt{\nu}\, \xi_B(t)$$

- Dissipation needed for stationarity
- Toy model of hydrodynamic equations (∞ stable states)

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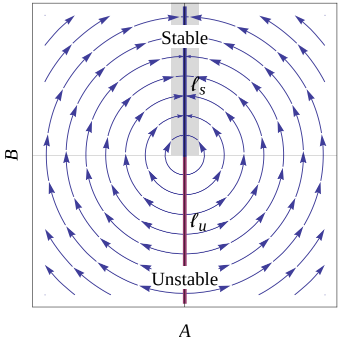

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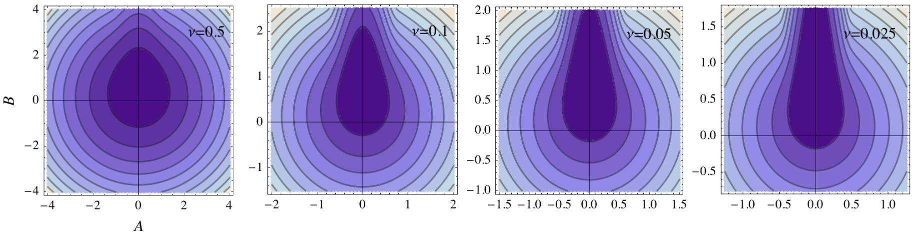
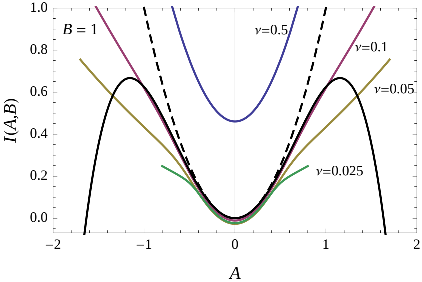
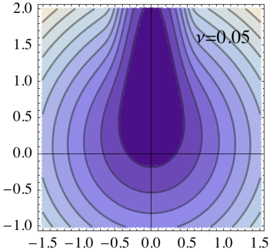


- $P(A,B)$
- Numerical integration of Fokker-Planck equation
- Concentration around stable line as $\nu \to 0$
- Radial symmetry away from stable line

Large deviations near stable line





- Stationary distribution:

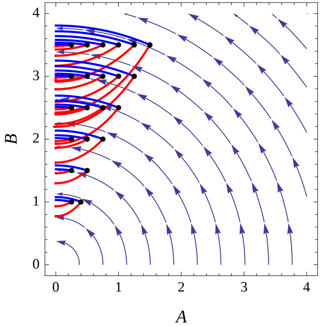
$$P(A,B) \sim e^{-I(A,B)/\nu}$$

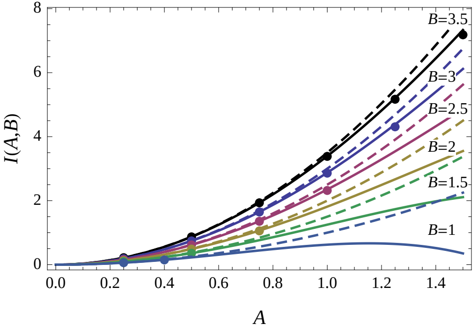
- Rate function or quasi-potential:

$$I(A,B) = \frac{B}{\sigma_A^2}A^2 - \frac{2\sigma_A^2+\sigma_B^2}{8\sigma_A^4 B}A^4 + O(A^6)$$

  - Instanton approximation = Fokker-Planck ν-expansion lowest order
  - Fokker-Planck ν-expansion – higher order

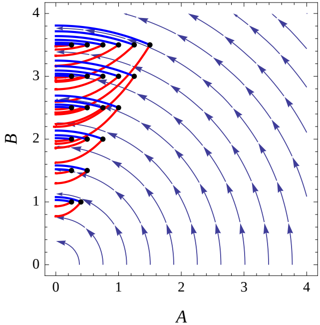
Large deviations near stable line (cont'd)

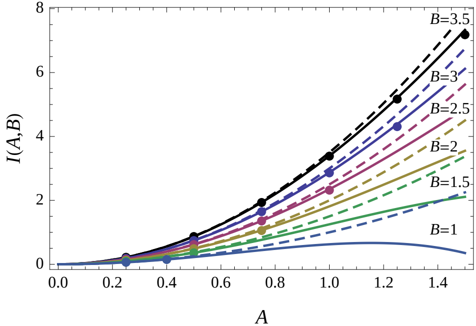




- Instanton: stable line $\longrightarrow (A, B)$
  - $I(A, B) = I[\textit{instanton}] > 0$
- Decay path: $(A, B) \longrightarrow$ stable line
  - $I[\textit{decay path}] = 0$
- Instanton $\neq$ Time reverse of decay path
- Nonequilibrium (non-gradient) system

Large deviations near stable line (cont'd)





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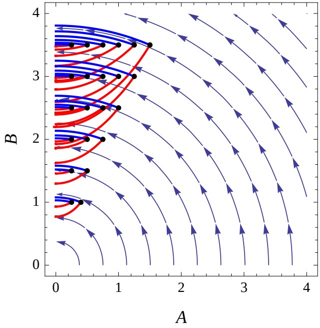

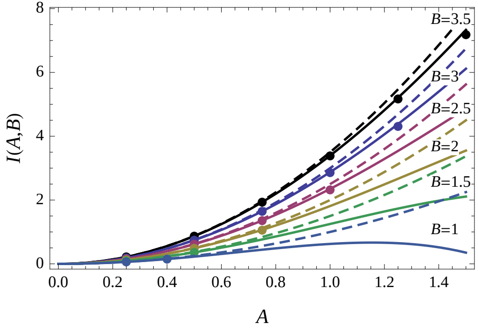

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Nonequilibrium current

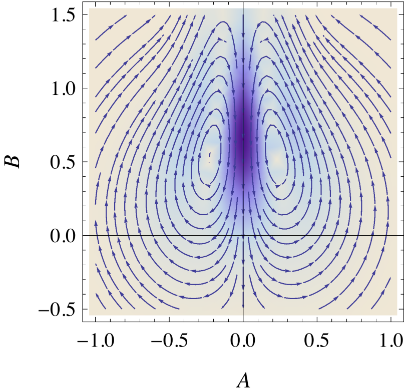
- Fokker-Planck equation:

$$\frac{\partial}{\partial t} P(A,B) = -\nabla \cdot \mathbf{J}$$

- Probability current:

$$\mathbf{J} = (J_A, J_B)$$

- Stationary current: $\nabla \cdot \mathbf{J} = 0$



- Components:

$$J_A = (-AB - \nu A)P(A,B) - \frac{\nu\sigma_A^2}{2}\frac{\partial P(A,B)}{\partial A}$$

$$J_B = (A^2 - \nu B)P(A,B) - \frac{\nu\sigma_B^2}{2}\frac{\partial P(A,B)}{\partial B}$$

Large deviations near unstable line

- Any point (A, B) reachable by instanton of zero action!
- Sub-instanton
- Consequence:

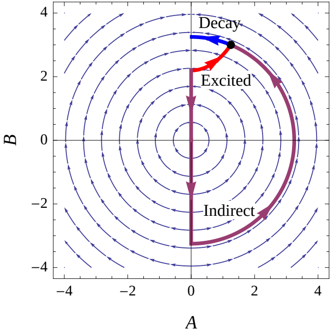
$$P(A,B) \sim e^{-0/\nu}$$

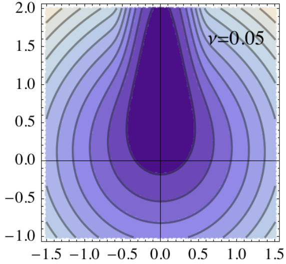
- Meaning:

$$P(A,B) \sim e^{-0/\nu} + \text{corrections}$$

- Competings large deviations:

$$P(A,B) \sim \underbrace{e^{-I(A,B)/\nu}}_{\text{stable line}} + \underbrace{e^{-J(A,B)/\sqrt{\nu}}}_{\text{unstable line}}$$





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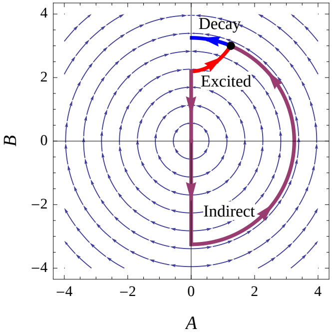
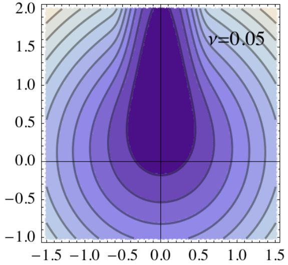




Large deviations near unstable line (cont'd)

- Low-noise expansion of Fokker-Planck equation
- Ansatz:

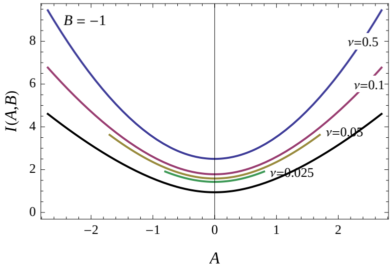
$$P(A,B) \sim e^{-J(A,B)/\sqrt{\nu}}$$

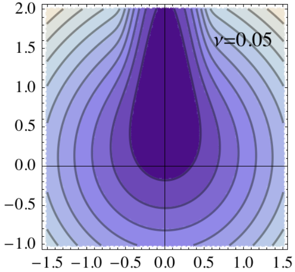
- Hamilton-Jacobi equation for $J(A,B)$
- Solve in polar coordinates
- Solution:

$$J(r) = \frac{2\sqrt{2}}{3} r^{3/2}$$

$$J(A,B) = \frac{2\sqrt{2}}{3}(A^2+B^2)^{3/4}$$

- Radially symmetric: Sub-instantons are radially symmetric





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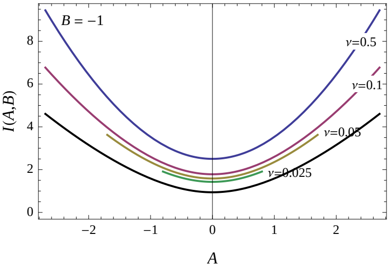

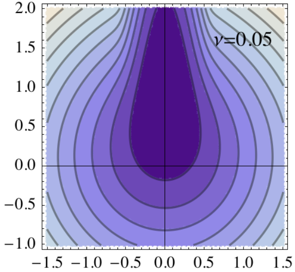

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Summary

- AB model: Nonequilibrium system
- Line of stable points connected to a line of unstable points
- Low-noise large deviations:

$$P(A,B) \sim \underbrace{e^{-I(A,B)/\nu}}_{\text{stable line}} + \underbrace{e^{-J(A,B)/\sqrt{\nu}}}_{\text{unstable line}}$$

- Explicit rate functions
  - Instanton approximation (Freidlin-Wentzell)
  - Low-noise expansion of Fokker-Planck
- Overall dominant term:

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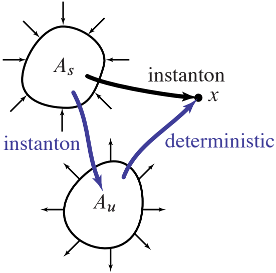
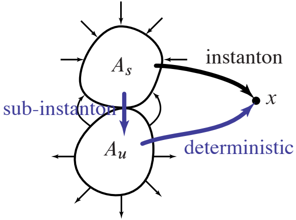
More general models





Unconnected sets

- All fluctuations paths are instantons
- $P(x) \sim e^{-I[\text{instanton}]/\nu}$
- Classical large deviations

Connected sets

- Instantons + sub-instantons
- $P(x) \sim e^{-I[\text{sub-instanton}]/\nu^{\alpha}}$
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- Exponent $\alpha = \frac{1}{2}$ always?
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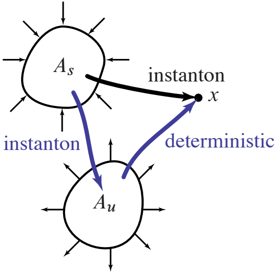

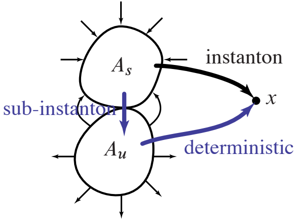

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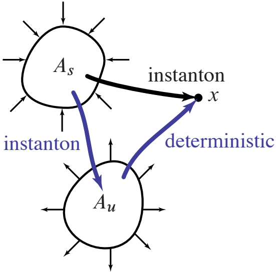

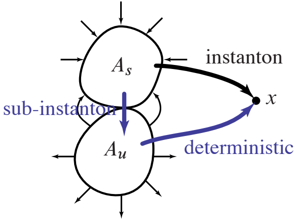

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