Fluctuations out of Equilibrium: Symmetries and Phase Transitions

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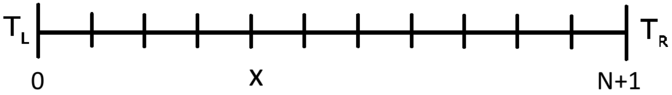
In collaboration with:
P. I. Hurtado, P. L. Garrido and J. del Pozo.

- Fluctuations arise universally in nature, playing a dominant role in many fields
- Encode fundamental information $\rightarrow$ Particularly important out of equilibrium
- Current statistics $\rightarrow$ Main objective of nonequilibrium statistical physics
- We focus on the study of current statistics in the 2D-KMP model

The KMP model in one dimension

C. kipnis, C. Marchioro and E. Presutti, Journal of Statistical Physics, **27** 65 (1982)

$e_x \in \Re_+$



e_x is interpreted as the energy of an oscillator at site x

- Stochastic dynamics: $e'_x = p(e_x + e_{x+1}) \quad e'_{x+1} = (1-p)(e_x + e_{x+1})$
- If $x = 0$ or $x = N+1$ we create a random $e_{0(N+1)}$ with the Gibbs distribution: $\beta_{L(R)} e^{-\beta_{L(R)} e_{0(N+1)}}$ with $\beta_{L(R)} = T_{L(R)}^{-1}$
- They proved that the system follows the Fourier's law with $\kappa[T] = \frac{1}{2}$

- P.I. Hurtado and P.L. Garrido studied the **Current Large Deviation** of the 1D-KMP model:

$$P(q_\tau, \tau; T_L, T_R) \underset{\tau \to \infty}{\simeq} \exp[\tau L G(q; T_L, T_R)] \qquad q_\tau = \frac{1}{\tau}\int_0^\tau dt \int_0^1 j(x,t)dx$$

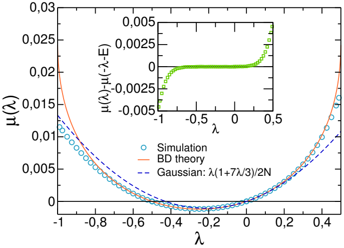
  The most probable value is the one for the stationary state:
  $G(\langle q\rangle; T_L, T_R) = 0; \quad G'(\langle q\rangle; T_L, T_R) = 0$

- They confirmed numerically the theoretical prediction (Additivity Principle) T. Bodineau and B. Derrida, PRL 92 180601 (2004)

$$G(q; T_L, T_R) = -\min_{T(x)}\left[\int_0^1 \frac{(q + \kappa[T(x)]\frac{dT}{dx})^2}{2\sigma[T(x)]}dx\right] \quad \text{KMP: } \kappa = \frac{1}{2}, \;\; \sigma[T] = T^2$$

Numerical results for the 1D-KMP model:

P.I. Hurtado and P.L. Garrido, PRL 102 250601 (2009)



$$\mu(\lambda) = \max_q[G(q; T_L, T_R) + \lambda q]$$

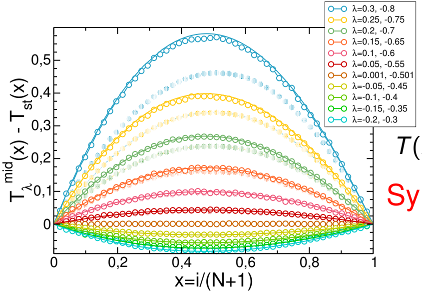
$$-\infty < q < \infty \Leftrightarrow -\frac{1}{T_R} < \lambda < \frac{1}{T_L}$$

$$T_L = 2 \qquad T_R = 1$$

The Gallavotti-Cohen theorem holds:

$$G(q; T_L, T_R) - G(-q; T_L, T_R) = q\left(\frac{1}{T_R} - \frac{1}{T_L}\right)$$

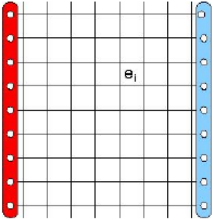
$$\mu(\lambda) = \mu\left(-\lambda - \frac{1}{T_R} + \frac{1}{T_L}\right)$$

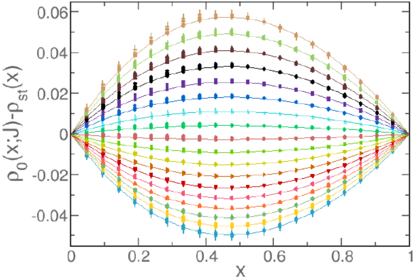


$$T(x; q) = T(x; -q) \Leftrightarrow T_\lambda(x) = T_{-\lambda - 1/T_R + 1/T_L}$$

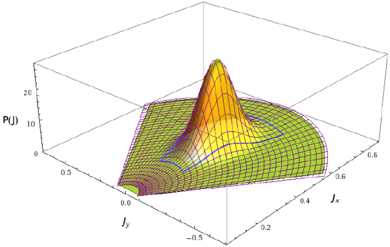
Symmetry due to the Gallavotti-Cohen Fluctuation Theorem

Temperature Profiles in the 2D-KMP model





$$\mathbf{J} = \frac{1}{\tau}\int_0^{\tau} dt \int_{\Lambda} d\mathbf{r}\,\mathbf{j}(\mathbf{r}, t)$$



Numerical evidence for the symmetry:

$$\boxed{T(\mathbf{r};\mathbf{J}) = T(\mathbf{r};|\mathbf{J}|)}$$

The Hydrodynamic Fluctuation Theory

L. Bertini,A. De Sole,D. Gabrielli,G. Jona-Lasisino,C. Landim

- We assume that our system is described by a Langevin type equation:

  $$\partial_t \rho(\mathbf{r}, t) = -\boldsymbol{\nabla} \cdot \mathbf{j}(\mathbf{r}, t) \qquad \text{with} \qquad \mathbf{j}(\mathbf{r}, t) = \mathbf{Q}[\rho] + \boldsymbol{\xi}(\mathbf{r}, t)$$

  where $\mathbf{r} \in \Lambda = [0, 1]^d$, $\boldsymbol{\xi}$ is a gaussian white noise and $\mathbf{Q}[\rho] = -D[\rho]\boldsymbol{\nabla}\rho$

- The probability to see a given space-time averaged value of the current is given by $P(\mathbf{J}) \simeq \exp[\tau L^d G(\mathbf{J})]$

$$G(\mathbf{J}) = -\frac{1}{\tau} \min_{\substack{\rho(\mathbf{r},t) \\ \mathbf{j}(\mathbf{r},t)}} \int_0^\tau dt \int_\Lambda d\mathbf{r} \frac{\Big(\mathbf{j}(\mathbf{r}, t) - \mathbf{Q}[\rho(\mathbf{r}, t)]\Big)^2}{2\sigma[\rho(\mathbf{r}, t)]}$$

- The minimization should be done with the conditions

  $$\rho(\mathbf{r}, t) = \bar{\rho}(\mathbf{r}; \mathbf{J}) \qquad \mathbf{j}(\mathbf{r}, t) = \mathbf{J}$$

Under all those conditions we find that

the fields that minimize the functional only depends on the modulus of the current:

$$\boxed{\bar{\rho}(\mathbf{r};\mathbf{J}) = \bar{\rho}(\mathbf{r};|\mathbf{J}|)}$$

The Isometric Fluctuation Relation (IFR)

The invariance of the most probable profile under current rotations implies ($\tau \to \infty$):

$$\boxed{\frac{P_\tau(\mathbf{J})}{P_\tau(S\mathbf{J})} \simeq \exp\left[\tau \boldsymbol{\epsilon} \cdot (\mathbf{J} - S\mathbf{J})\right]}$$

for any rotation S.

$\boldsymbol{\epsilon}$ only depends on boundary conditions. In particular $|\boldsymbol{\epsilon}|$=0 for systems in equilibrium.

Some Consequences of the IFR:

- If S is a rotation π the Gallavotti-Cohen FT holds:

$$\frac{P(\mathbf{J}_\tau = J)}{P(S\mathbf{J}_\tau = -J)} \simeq \exp[2\tau\epsilon J]$$

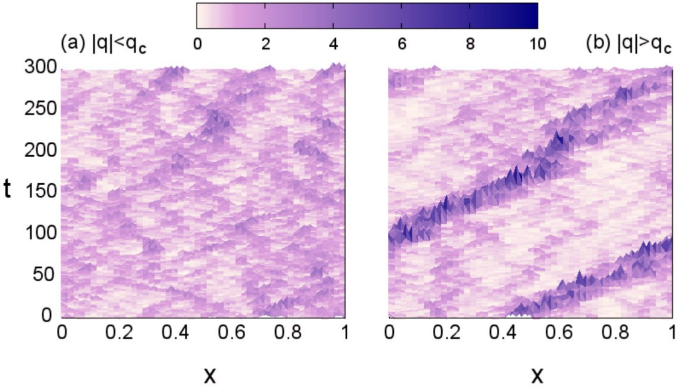
- The IFR implies relations between the cumulants of the current distribution

P.I. Hurtado,C. Pérez-Espigares,J. del Pozo and P.L. Garrido, PNAS 108, 7704 (2011)

Dynamical Phase Transition

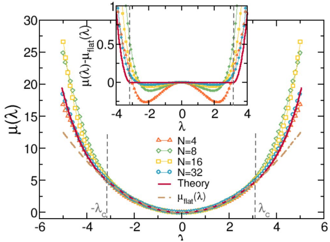
- The HFT predicts in general a time-dependent optimal path in phase space responsible of given current fluctuation (Bertini et al., Bodineau & Derrida)
- Additivity conjecture: optimal path is time-independent in a broad regime
- This scenario eventually breaks down for large fluctuations via a dynamic phase transition at the fluctuating level
- Hurtado and Garrido observed this for current fluctuations in 1D KMP energy-diffusion model on a ring

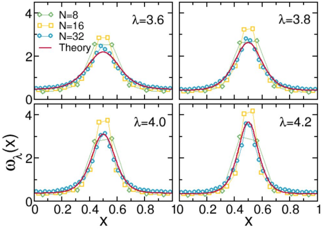
P.I. Hurtado, P.L. Garrido, PRL 107, 180601 (2011)

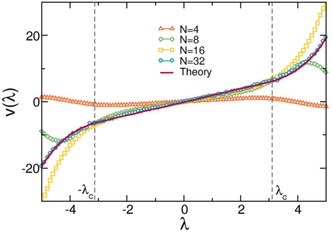


- $|q| < q_c$ → sum of weakly-correlated local events → Gaussian stat.
- $|q| > q_c$ → coherent traveling wave + energy localization → non-Gaussian
- Striking phenomenon: isolated equilibrium system with no external fields

Again they studied the current large deviation:







- Agreement with the HFT is very good for large enough N at the fluctuation level

- The phase transition seems continuous as conjectured by Bodineau and Derrida, excluding the possibility of a first-order scenario

What happens in the 2D-KMP? (preliminary work)

Again we have

$$G(\mathbf{J}) = -\frac{1}{\tau} \min_{\substack{\rho(\mathbf{r},t) \\ \mathbf{j}(\mathbf{r},t)}} \int_0^\tau dt \int_\Lambda d\mathbf{r} \frac{\left(\mathbf{j}(\mathbf{r},t) - \mathbf{Q}[\rho(\mathbf{r},t)]\right)^2}{2\sigma[\rho(\mathbf{r},t)]}$$

with $\mathbf{J} = \frac{1}{\tau}\int_0^\tau dt \int_\Lambda \mathbf{j}(\mathbf{r},t) d\mathbf{r}$, $\quad \frac{\partial \rho(\mathbf{r},t)}{\partial t} = -\nabla \cdot \mathbf{j}(\mathbf{r},t)$, $\quad \int_\Lambda \rho(\mathbf{r},t) d\mathbf{r} = \rho_0$

We propose a time dependent solution $\rho(\mathbf{r},t) = \omega(\mathbf{r} - \mathbf{v}t)$

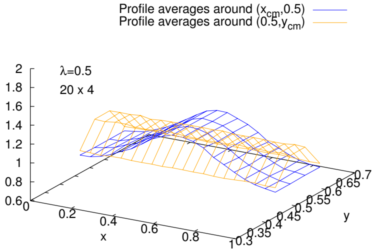
$$G(\mathbf{J}) = -\min_{\mathbf{v},\omega} \int_\Lambda d\mathbf{r} \frac{(\mathbf{v}\omega(\mathbf{r}) + \mathbf{J} - \mathbf{v}\rho_0 + D[\omega]\nabla\omega(\mathbf{r}))^2}{2\sigma[\omega]}$$

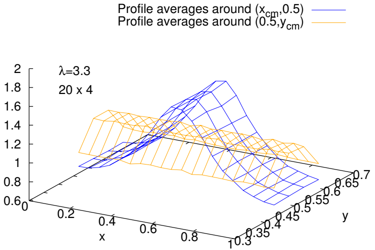
$$\Rightarrow \boxed{(\nabla\omega)^2 = \frac{1}{D[\omega]^2}\left[(\mathbf{J} - \mathbf{v}\rho_0 + \mathbf{v}\omega(\mathbf{r}))^2 - C_2\omega 2\sigma[\omega] - C_1 2\sigma[\omega]\right]}$$

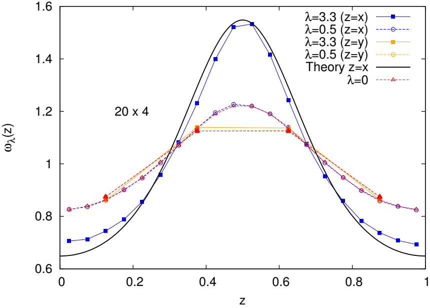
We have many possible solutions... Which one do we study?

Let's see simulations!

We could simulate very moderate aspect ratio ($\alpha = L_x \times L_y = 20 \times 4 = 0.2$) $\boldsymbol{\lambda} = (\lambda_x, 0)$







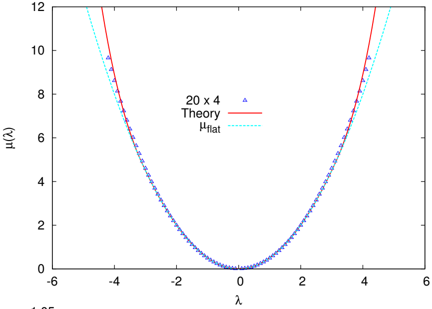
- The profile only changes in the x-direction

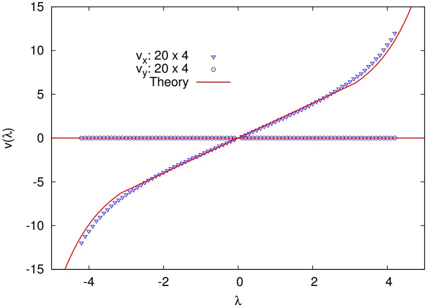
  increasing $\boldsymbol{\lambda}$

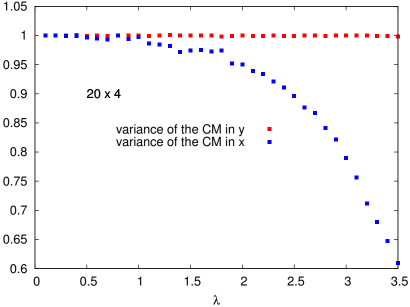
- We study the wave solution $\omega(\mathbf{r}) = \omega(x)$
- The profiles are the same of the 1D system
- The velocities too

$$\mu_{2D}(\boldsymbol{\lambda}) = \frac{\mu_{1D}(|\boldsymbol{\lambda}|)}{\alpha}$$

Legendre transform of the large deviation and velocity of the wave







← The variance of the CM in each direction

could be an order parameter

Conclusions:

- For a moderate aspect ratio we have a dynamical phase transition
- It is a travelling wave
- Suggest that a traveling wave is in fact the most favorable time-dependent profile in the supercritical regime
- Rare events call in general for coherent, self-organized patterns in order to be sustained
- What happens for an aspect ratio $\alpha = 1$? Are other solutions more probable?

Thank you !