Long-range correlations in driven systems

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Driven, non-equilibrium systems

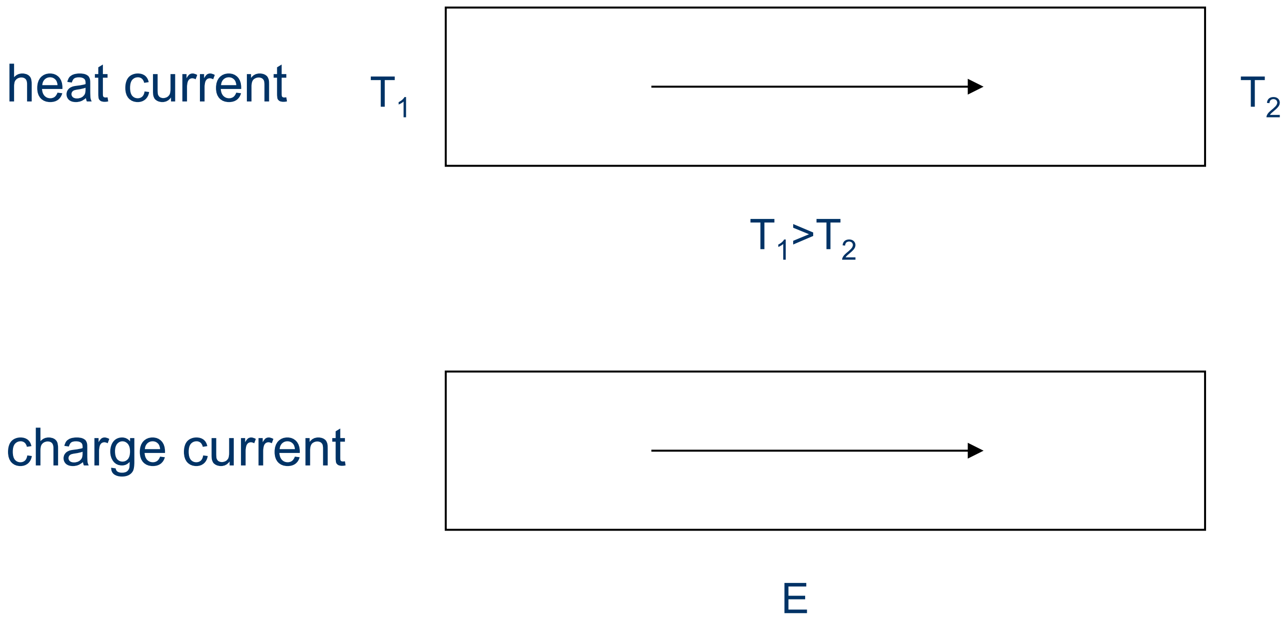
- systems with currents
- do not obey detailed balance

$P(C)$ the probability to be at a microscopic configuration C at time t

$$\frac{\partial P(C)}{\partial t} = \sum_{C'} W(C' \to C)P(C') - \sum_{C'} W(C \to C')P(C)$$

$$W(C' \to C)P(C') = W(C \to C')P(C)$$

Typical simple examples



What are the steady state properties of such systems?

It is well known that such systems exhibit long-range correlations when the dynamics involves some conserved parameter.

Outline

Will discuss a few examples where long-range correlations show up and consider some consequences

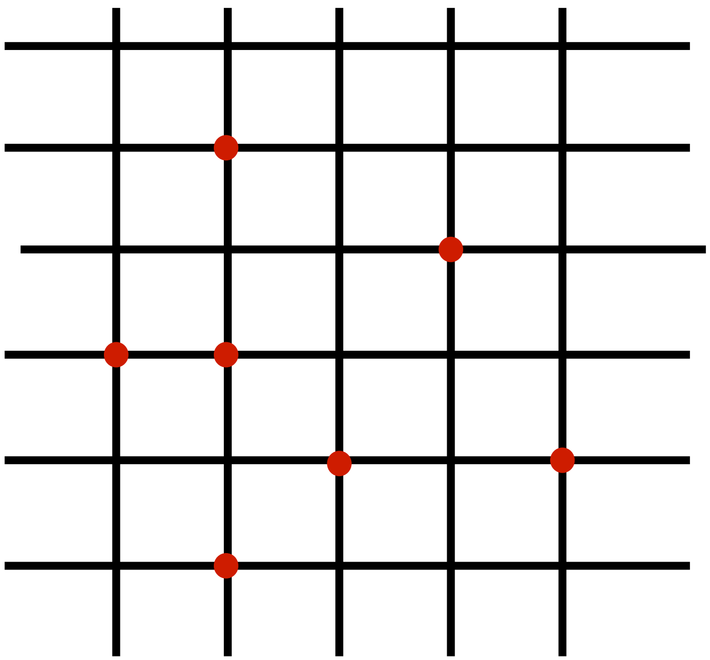
- Example I: Effect of a local drive on the steady state of a system
- Example II: Linear drive in two dimensions: spontaneous symmetry breaking

- Example I :Local drive perturbation

T. Sadhu, S. Majumdar, DM, Phys. Rev. E 84, 051136 (2011)

Local perturbation in equilibrium

Particles diffusing (with exclusion) on a grid



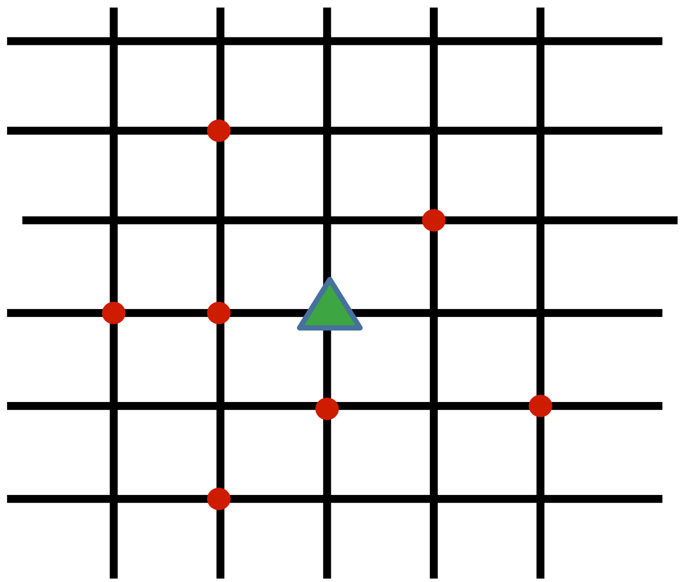
occupation number $\tau_i = 0{,}1$

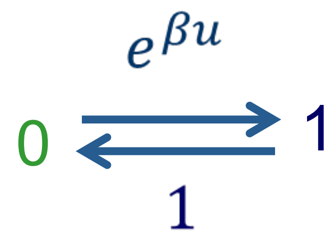
N particles
V sites

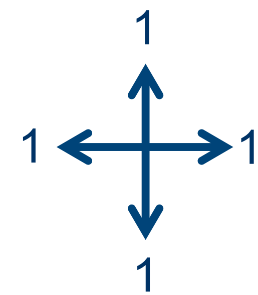
Prob. of finding a particle at site k $\quad p(k) = \dfrac{N}{V}$

Add a local potential u at site 0

N particles
V sites





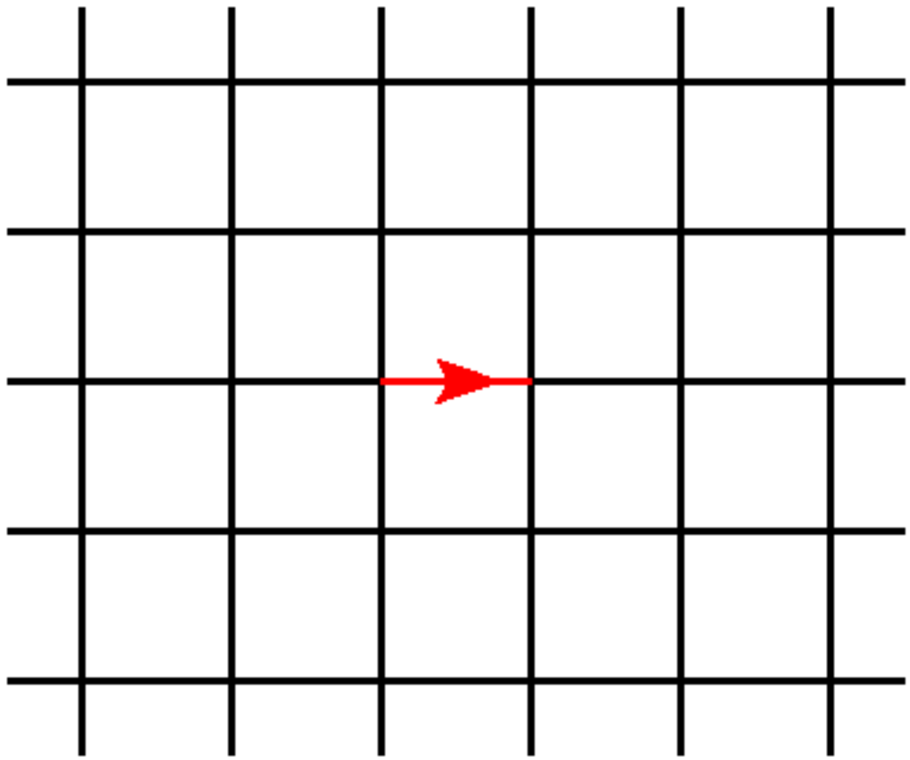


$$(V-1)P(k) + P(0) = N \qquad (k \neq 0)$$

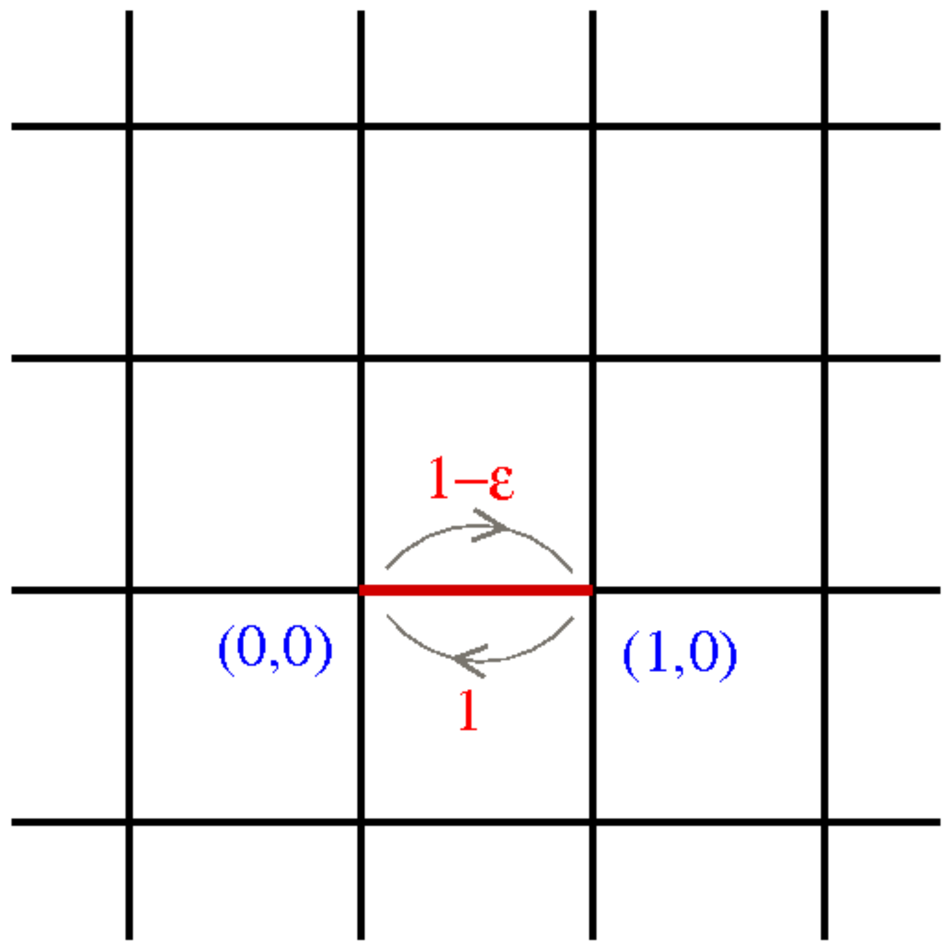
$$P(k) = \frac{N - P(0)}{V-1} \approx \frac{N}{V} + O(\frac{1}{V})$$

The density changes only locally.

How does a local drive affect the steady-state of a system?



A single driving bond



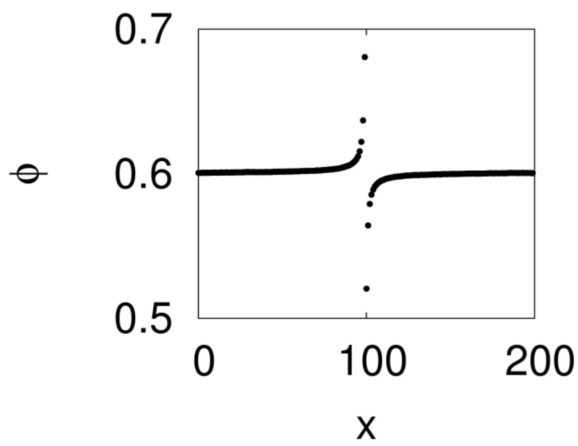
Main results:

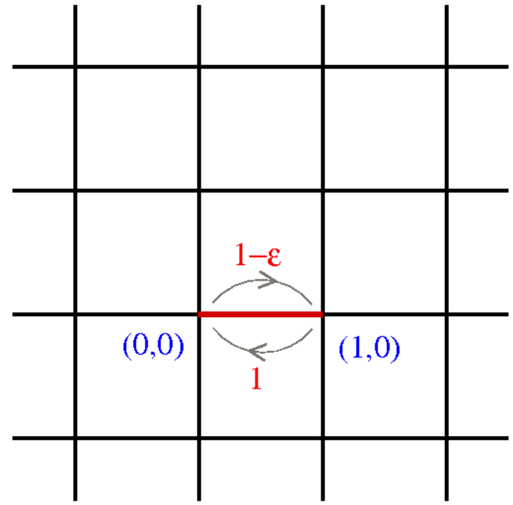
- In d ≥ 2 dimensions both the density and the local current decay algebraically with the distance from the driven bond.

- The same is true for local arrangements of driven bond. The power law of the decay depends on the specific configuration.

- In d=2 dimensions a close correspondence to electrostatics is found, with analogous variables to electric and magnetic fields E, H.

Density profile (with exclusion)





The density profile $\phi(\vec{r}) \sim \begin{cases} 1/r^2 & \text{along the y axis} \\ 1/r & \text{in any other direction} \end{cases}$

Non-interacting particles

- Time evolution of density:

$$\partial_t \phi(\vec{r},t) = \nabla^2(\vec{r},t) + \epsilon\phi\left(\vec{0},t\right)[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\overrightarrow{e_1}}]$$

$$\nabla^2 = \phi(m+1,n) + \phi(m-1,n) + \phi(m,n+1) + \phi(m,n-1) - 4\phi(m,n)$$

The steady state equation

$$\nabla^2\phi(\vec{r}) = -\epsilon\phi(\vec{0})[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\overrightarrow{e_1}}]$$

particle density → electrostatic potential of an electric dipole

$$\nabla^2 \phi(\vec{r}) = -\epsilon \phi(\vec{0})[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\overrightarrow{e_1}}]$$

The dipole strength has to be determined self consistently.

Green's function $$\nabla^2 G(\vec{r}, \vec{r}_o) = -\delta_{\vec{r},\vec{r}_o}$$

solution $$\phi(\vec{r}) = \rho + \epsilon \phi(\vec{0})[G(\vec{r}, \vec{0}) - G(\vec{r}, \vec{e}_1)]$$

Unlike electrostatic configuration here the strength of the dipole should be determined self consistently.

Green’s function of the discrete Laplace equation

p\q	0	1	2
0	0	$-\frac{1}{4}$	$\frac{2}{\pi}-1$
1	$-\frac{1}{4}$	$-\frac{1}{\pi}$	$-\frac{1}{4}$
2	$\frac{2}{\pi}-1$	$\frac{1}{4}-\frac{2}{\pi}$	$-\frac{4}{3\pi}$

$$G(\vec{r},\vec{r}_o) \approx -\frac{1}{2\,\pi} \ln|\vec{r}-\overrightarrow{r_0}|$$

$$\phi(\vec{r}) = \rho + \epsilon\phi(\vec{0})[G(\vec{r},\vec{0}) - G(\vec{r},\vec{e}_1)]$$

determining $\phi(\vec{0})$

$$\phi(\vec{r}) = \rho + \epsilon\phi(\vec{0})[G(\vec{r},\vec{0}) - G(\vec{r},\vec{e}_1)]$$

To find $\phi(\vec{0})$ one uses the values $G(\vec{0},\vec{0}) = 0$, $G(\vec{0},\vec{e}_1) = -\frac{1}{4}$

$$\boxed{\phi(\vec{0}) = \frac{\rho}{1-\frac{\epsilon}{4}}}$$

at large $\vec{r}$

$$G(\vec{r},\vec{r}_o) \approx -\frac{1}{2\,\pi}\,\ln|\vec{r}-\overrightarrow{r_0}|$$

$$\phi(\vec{r}) = \rho + \epsilon\phi\left(\vec{0}\right)\left[G\left(\vec{r},\vec{0}\right) - G(\vec{r},\vec{e}_1)\right]$$

$$\phi(\vec{0}) = \frac{\rho}{1-\frac{\epsilon}{4}}$$

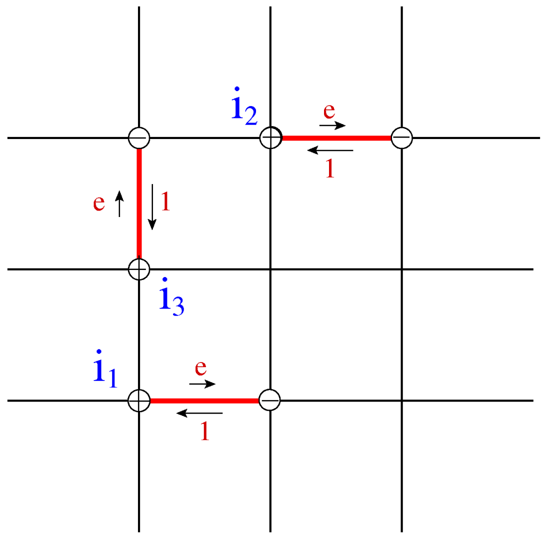
density:

$$\phi(\vec{r}) = \rho - \frac{\epsilon\phi\left(\vec{0}\right)}{2\pi}\frac{\vec{e}_1\vec{r}}{r^2} + O(\frac{1}{r^2})$$

current:

$$j(\vec{r}) = \frac{\epsilon\phi\left(\vec{0}\right)}{2\pi}\frac{1}{r^2}[\vec{e}_1 - \frac{2(\vec{e}_1\vec{r})\vec{r}}{r^2} + O(\frac{1}{r^3})$$

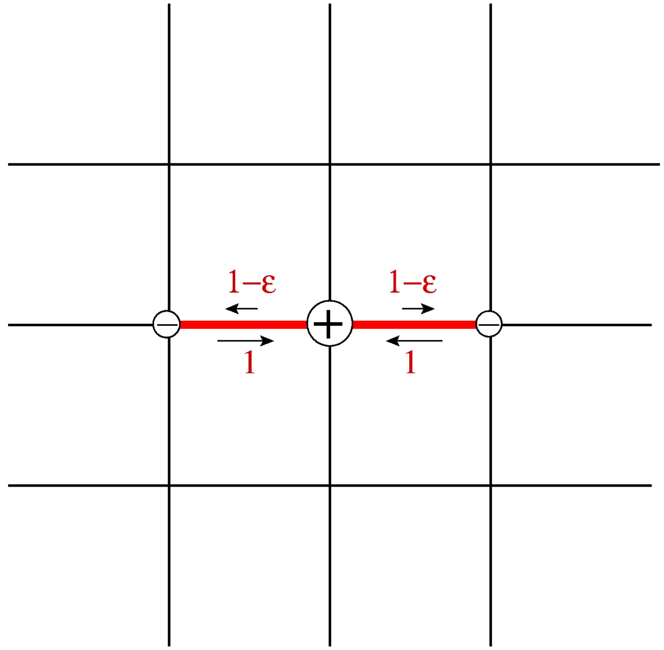
Multiple driven bonds



$$\phi(\vec{r}) = \rho + \epsilon\phi(\vec{\imath}_1)[G(\vec{r},\vec{\imath}_1) - G(\vec{r},\vec{\imath}_1 + \vec{e}_1)] + \epsilon\phi(\vec{\imath}_2)[G(\vec{r},\vec{\imath}_2) - G(\vec{r},\vec{\imath}_2 + \vec{e}_1)] + \cdots$$

Using the Green's function one can solve for $\phi(\vec{\imath}_1)$, $\phi(\vec{\imath}_2)$…
by solving the set of linear equations for $\vec{r} = \overrightarrow{i_1}_1, \overrightarrow{i_2}, \cdots$

Two oppositely directed driven bonds – quadrupole field



The steady state equation: $\nabla^2\phi(\vec{r}) = -\epsilon\phi(\vec{0})[2\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e}_1} - \delta_{\vec{r},-\vec{e}_1}]$

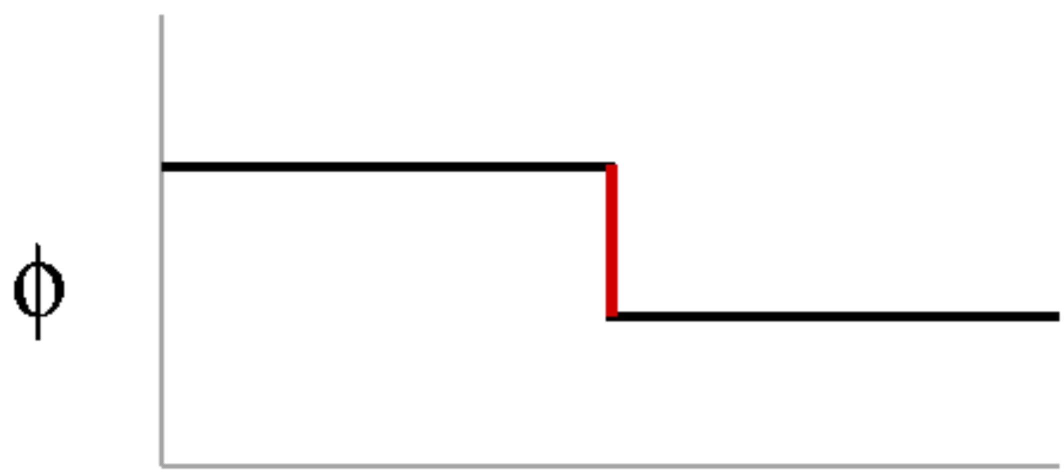
$$\phi(\vec{r}) = \rho - \frac{\epsilon\phi(\vec{0})}{2\pi}\left[\frac{1}{r^2} - 2\left(\frac{\vec{e}_1\vec{r}}{r^2}\right)^2\right] + O(\frac{1}{r^4})$$

$d \neq 2$ dimensions

$d \geq 2$ $\qquad \phi(\vec{r}) \sim \frac{1}{r^{d-1}}$

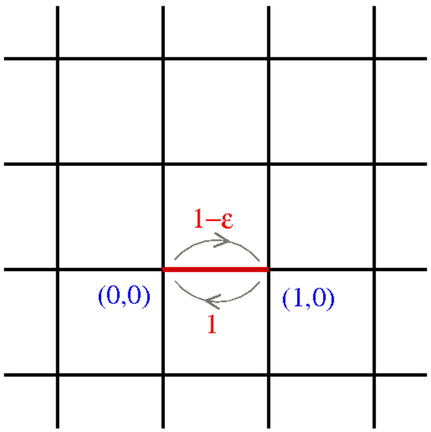
$d = 1$ $\qquad \phi(x) = \rho - \left(\frac{\epsilon}{2}\right)\phi(0)sgn(x)$

$$G(x, x_o) = -\frac{|x - x_o|}{2}$$



The model of local drive with exclusion

Here the steady state measure is not known however one can determine the behavior of the density.



$$\partial_t \phi(\vec{r},t) = \nabla^2(\vec{r},t) + \epsilon\langle\tau(\vec{0})\{1-\tau(\vec{e}_1)\}\rangle[\delta_{\vec{r},\vec{0}} - \delta_{\vec{r},\vec{e_1}}]$$

$\tau = 0,1$ is the occupation variable

$$\phi(\vec{r}) = \rho - \frac{\epsilon\langle\tau(\vec{0})\{1-\tau(\vec{e}_1)\}\rangle}{2\pi}\frac{\vec{e}_1\vec{r}}{r^2} + O(\frac{1}{r^2})$$

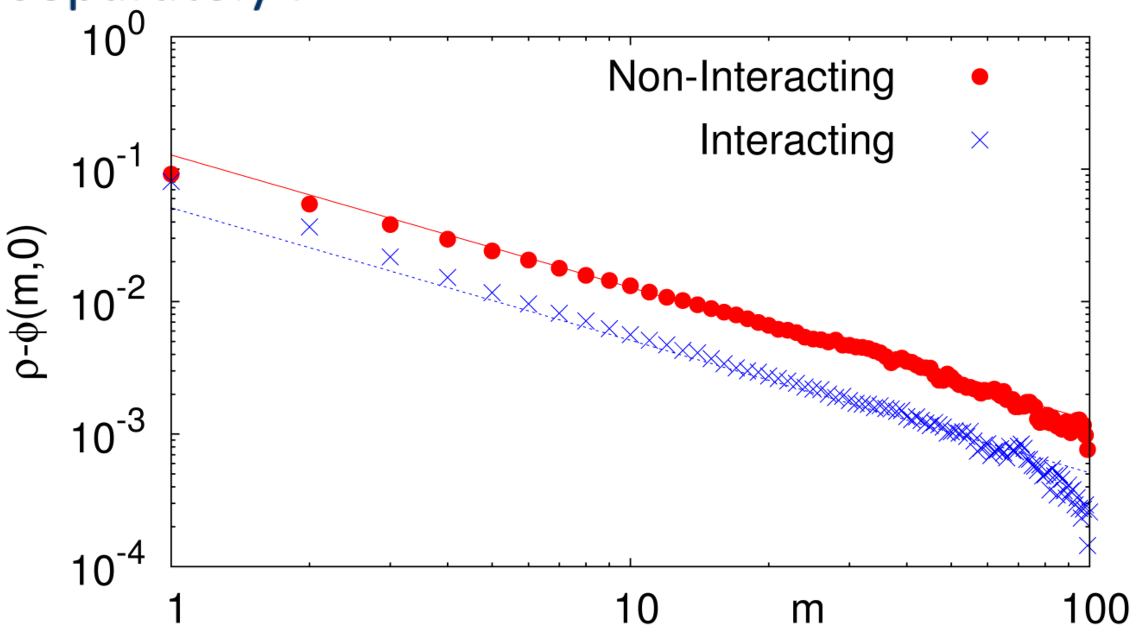
$$\phi(\vec{r}) = \rho - \frac{\epsilon\langle\tau(\vec{0})\{1-\tau(\vec{e}_1)\}\rangle}{2\pi}\frac{\vec{e}_1\vec{r}}{r^2} + O(\frac{1}{r^2})$$

The density profile is that of the dipole potential with a dipole strength which can only be computed numerically.

Simulation results

Simulation on a 200×200 lattice with $\rho = 0.6$

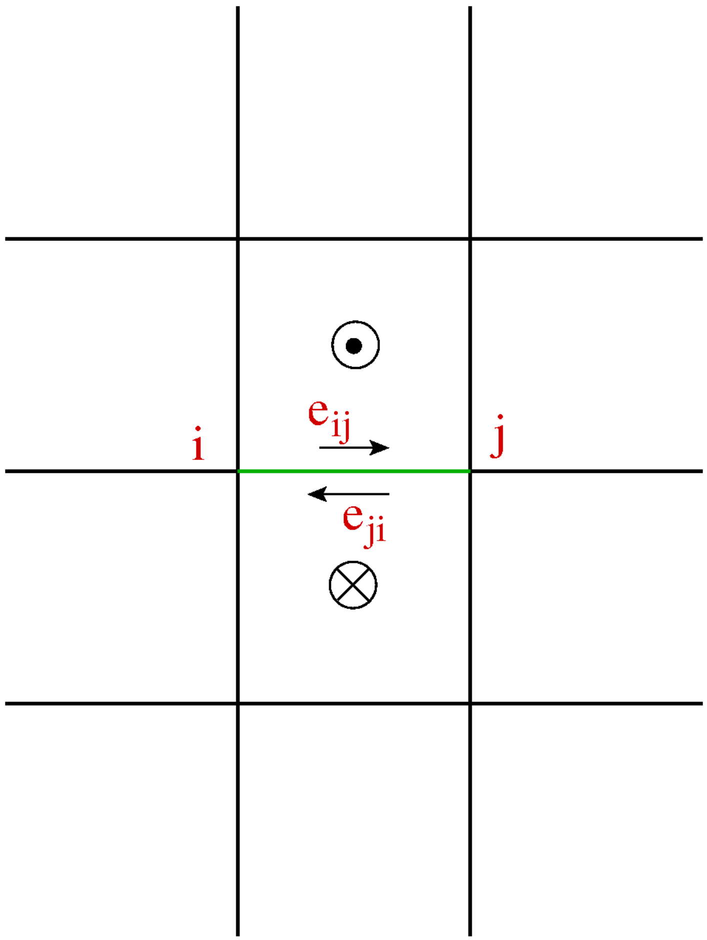
For the non-interacting case strength of the dipole was measured separately .



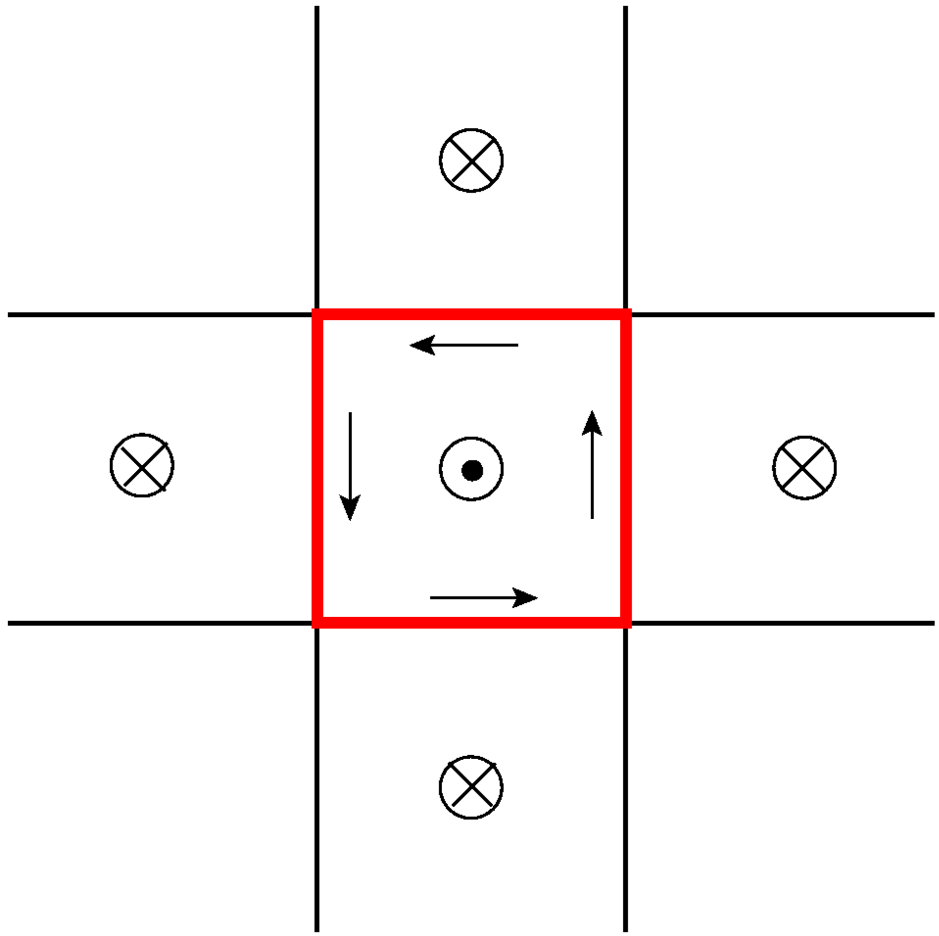
Magnetic field analog

for $i \to j$ process $H = \ln[e_{i,j}]$

for the i, j bond $H = \ln \frac{e_{ij}}{e_{ji}}$

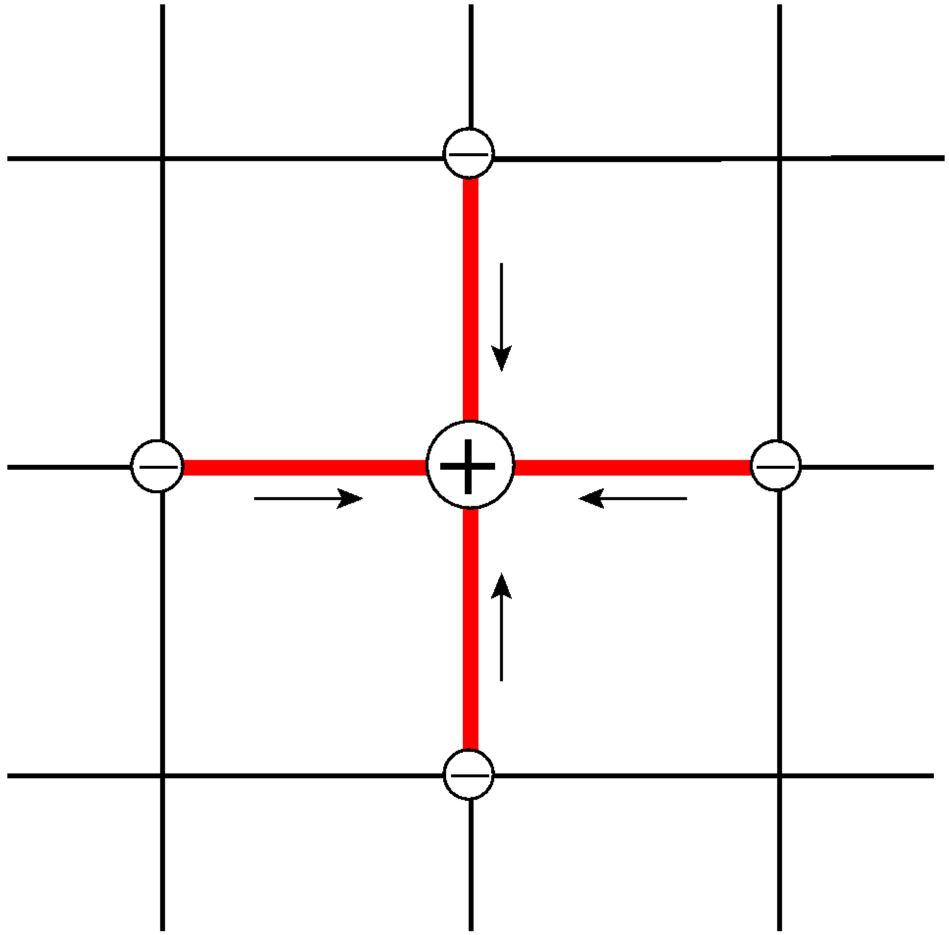


Zero-charge configuration



The density is flat however there are currents

Zero magnetic field configuration



no currents but inhomogeneous density (equilibrium)

In general

- non-zero electric field → inhomogeneous density
- non-zero magnetic field → currents
- zero magnetic field → equilibrium configuration

- Example II: a two dimensional model with a driven line

T. Sadhu, Z. Shapira, DM

Two dimensional lattice gas (Ising) model (equilibrium)

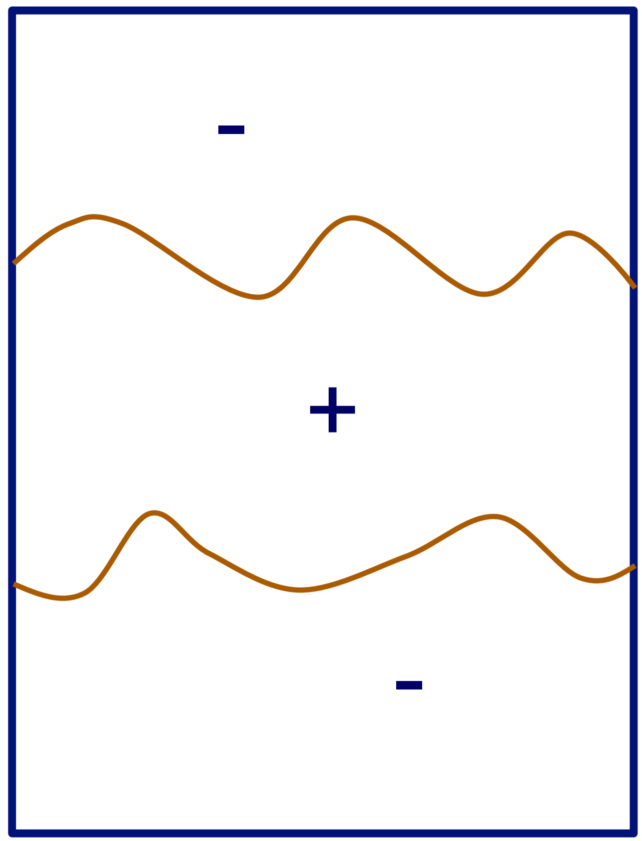
$$H = -J \sum_{<ij>} S_i S_j \qquad\qquad S_i = \pm 1$$

+ particle   - vacancy

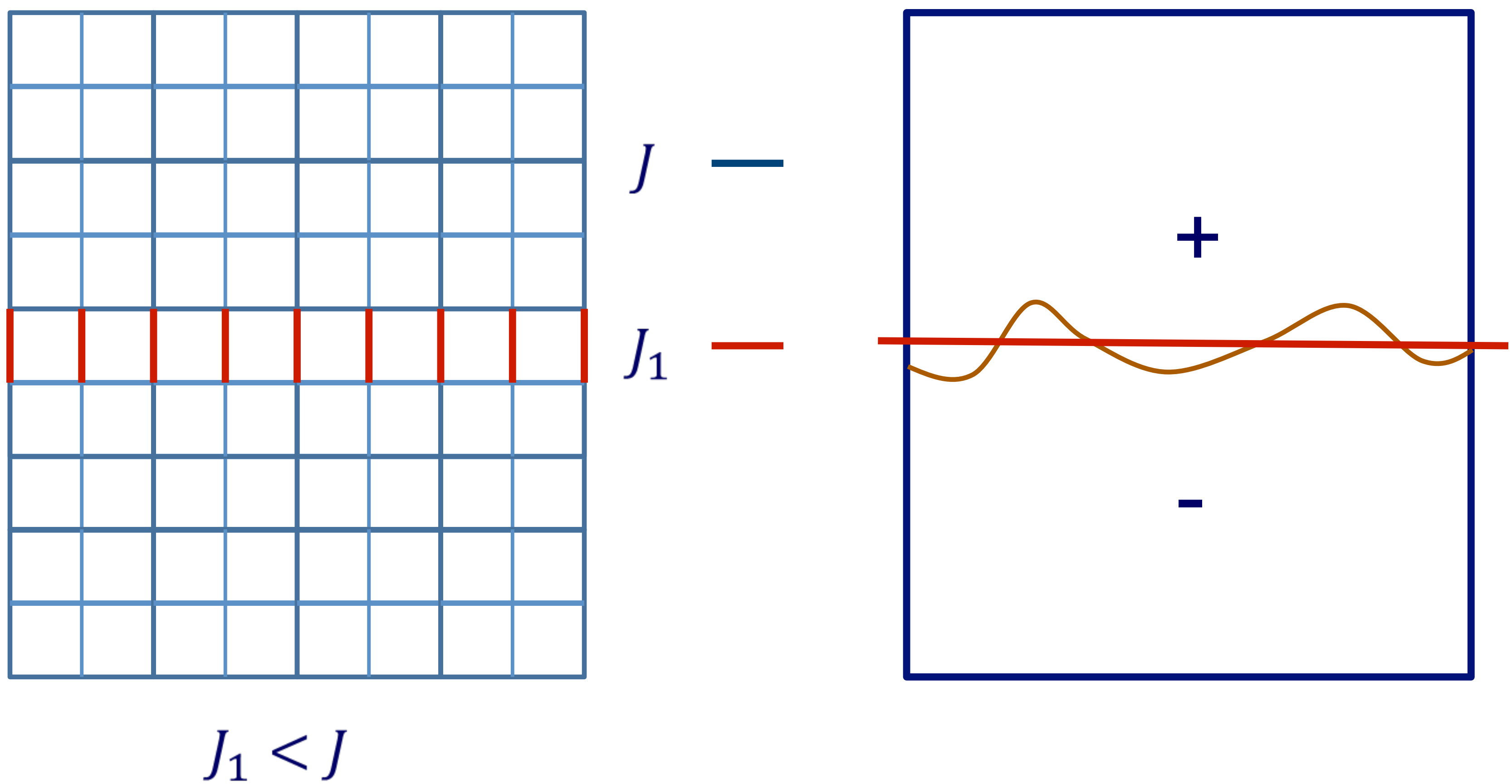
particle exchange (Kawasaki) dynamics

$+ - \rightarrow - +$ with rate $\min(1,\, e^{-\beta \Delta H})$

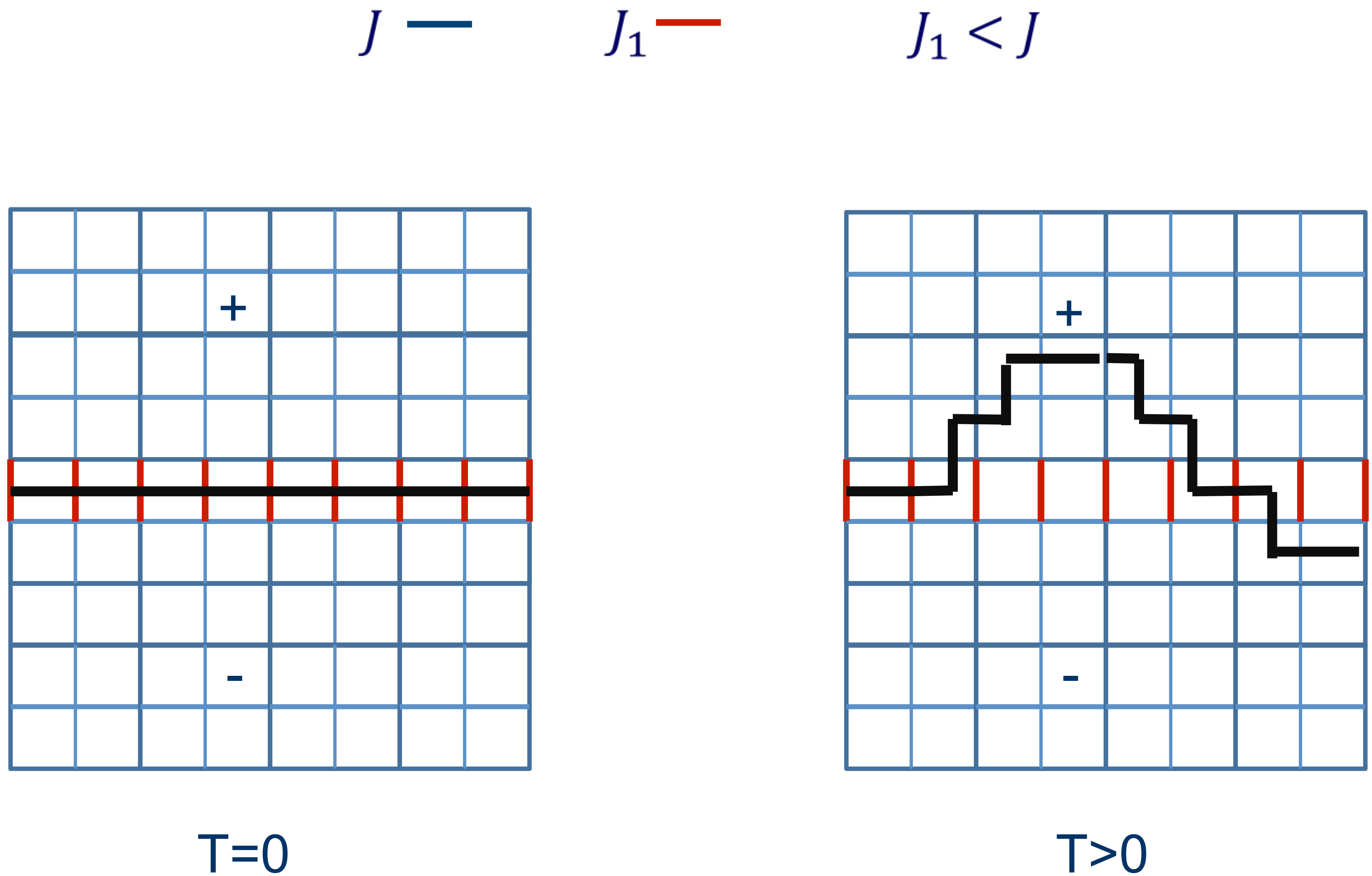
at $T < T_c$



2d Ising model with a row of weak bonds (equilibrium)



The weak-bonds row localizes the interface at any temperature $T < T_c$



J
J1
J1 < J
+
-
T=0
+
-
T>0

Interface energy at low temperature:

$$H = J \sum_{i=1}^{L} |h_i - h_{i+1}| - (J - J_1) \sum_{i=1}^{L} \delta(h_i)$$

$P(h)$ - the probability of finding the interface at height h

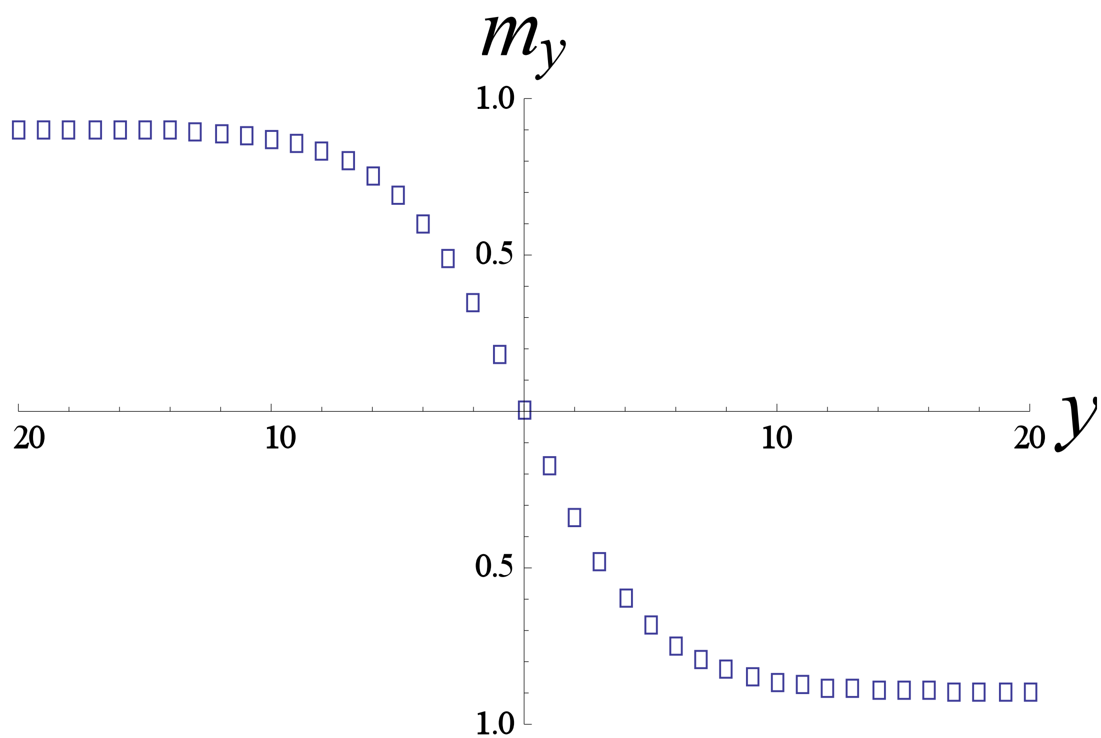
$$P(h+1) + P(h-1) - 2P(h) = -\lambda P(h) \quad h \neq 0$$

$$P(L) + P(1) - 2P(0) = (-\lambda - \epsilon)P(0)$$

$$\epsilon = -\left(e^{\beta(J-J_1)} - 1\right)e^{\beta} \quad (< 0)$$

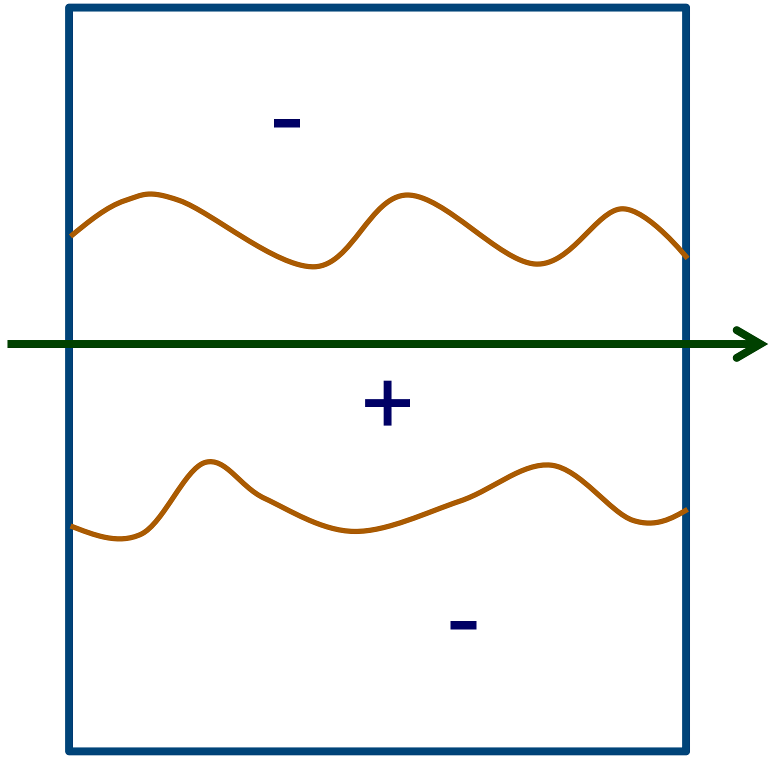
1d Quantum mechanical particle (discrete space) with a local attractive potential. The wave function is localized.

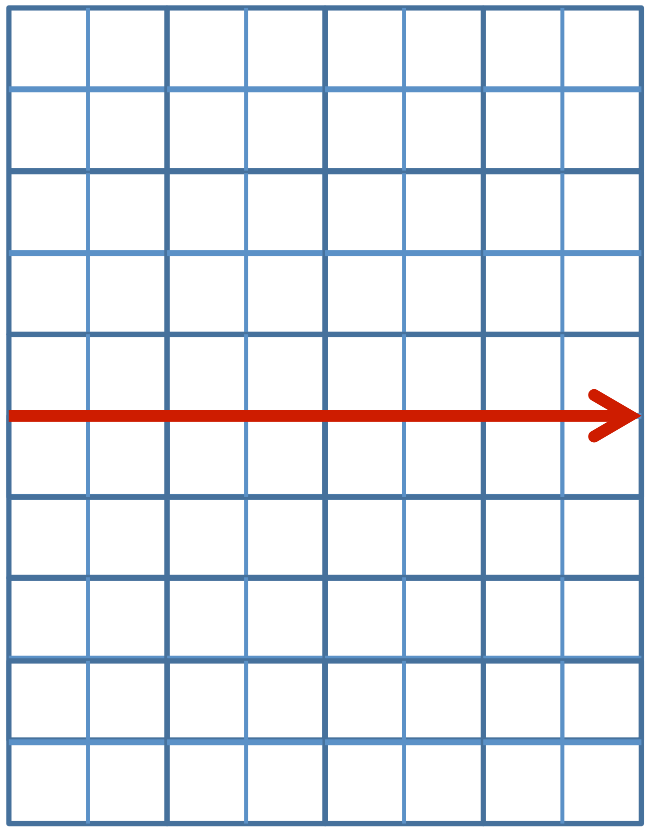
Schematic magnetization profile



The magnetization profile is antisymmetric with respect to the zero line with $m(y = 0) = 0$

Consider now a line of driven bonds





+ - → - +   with rate $\min(1, e^{-\beta\Delta H+\beta E})$

- + → + -   with rate $\min(1, e^{-\beta\Delta H-\beta E})$

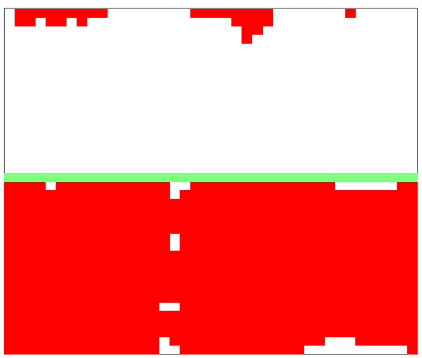
$$H = -J \sum_{<ij>} S_i S_j$$

Main results

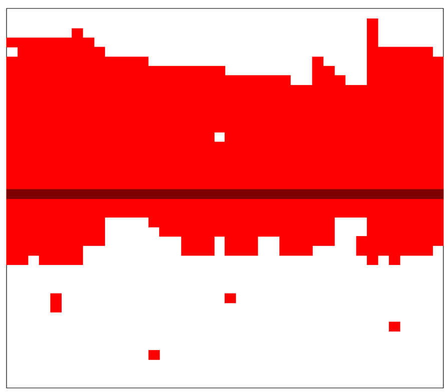
- The driven line attracts the interface
- The interface width is finite (localized)
- A spontaneous symmetry breaking takes place by which the magnetization of the driven line is non-zero and the magnetization profile is not antisymmetric, (mesoscopic transition).
- The fluctuation of the interface are not symmetric around the driven line.
- These results can be demonstrated analytically in certain limit.

Results of numerical studies

The is attracted by the driven line.

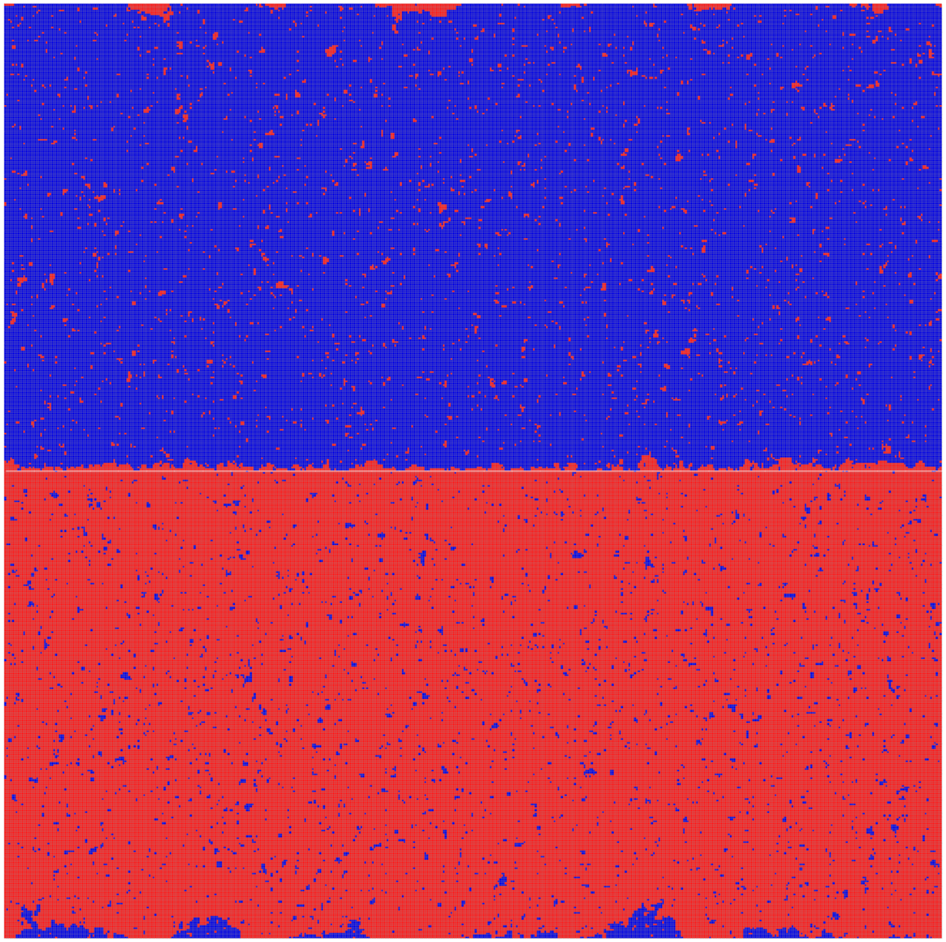


Time= 3e9



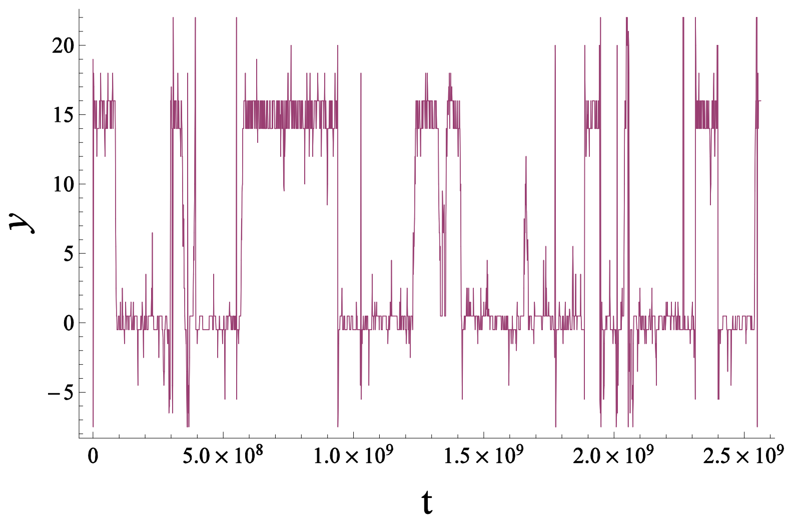
Time= 5e9

Time= 6e9

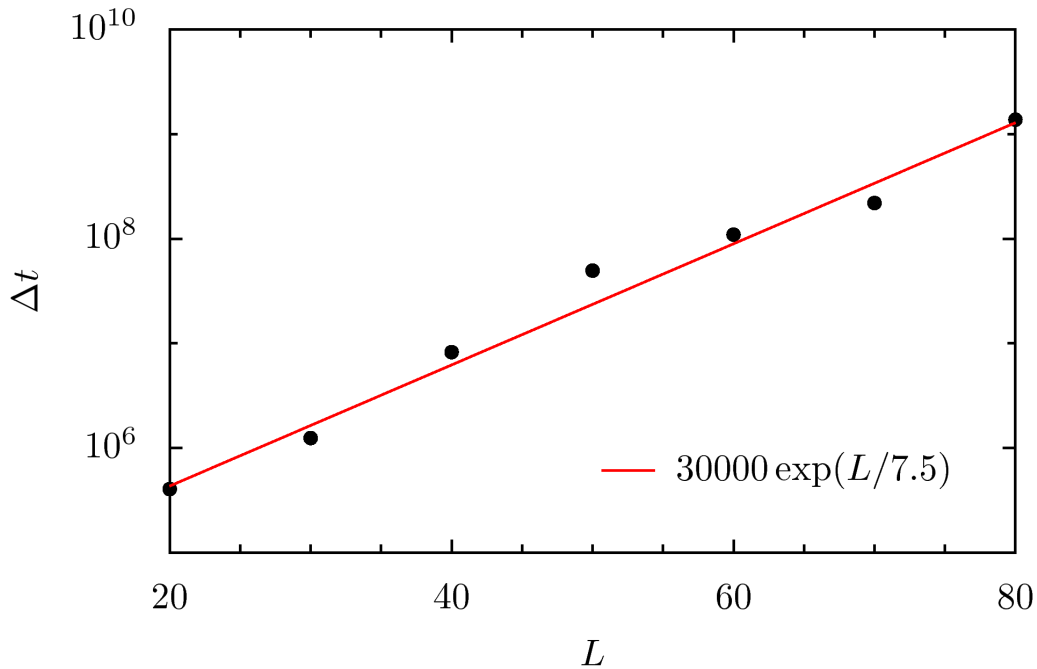


A configuration of the periodic 500X501 lattice at temperature 0.85Tc.

Temporal evolution of the interface position

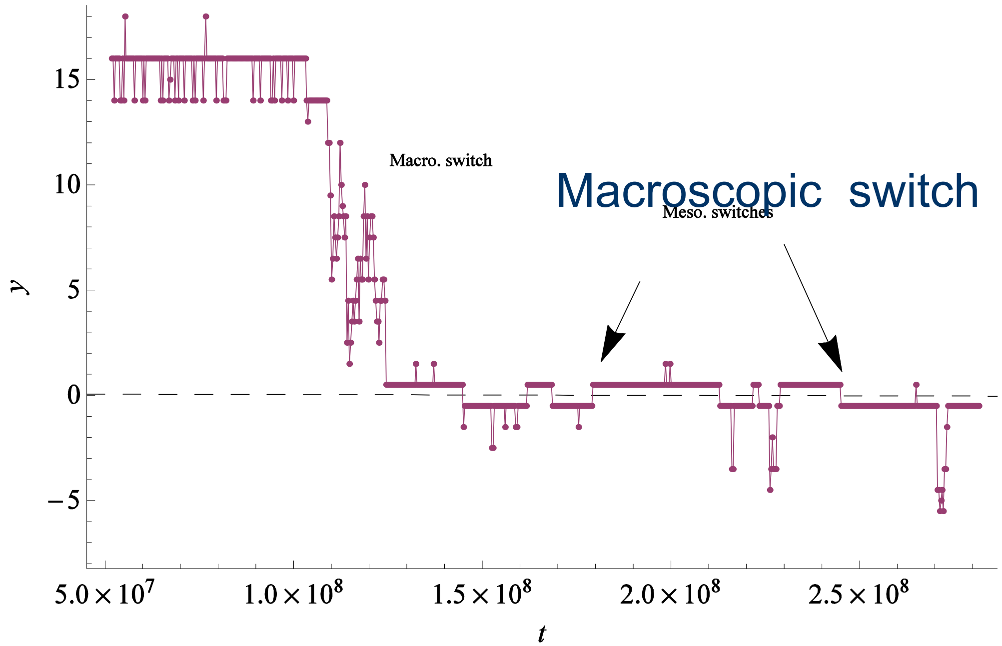


Periodic 30X31 lattice at temperature 0.6Tc. Driven lane at y=0, there are around 15 macro-switches on a 10^9 MC steps.



10^{10}
10^8
10^6
Δt
20
40
60
80
L
$30000\,\exp(L/7.5)$

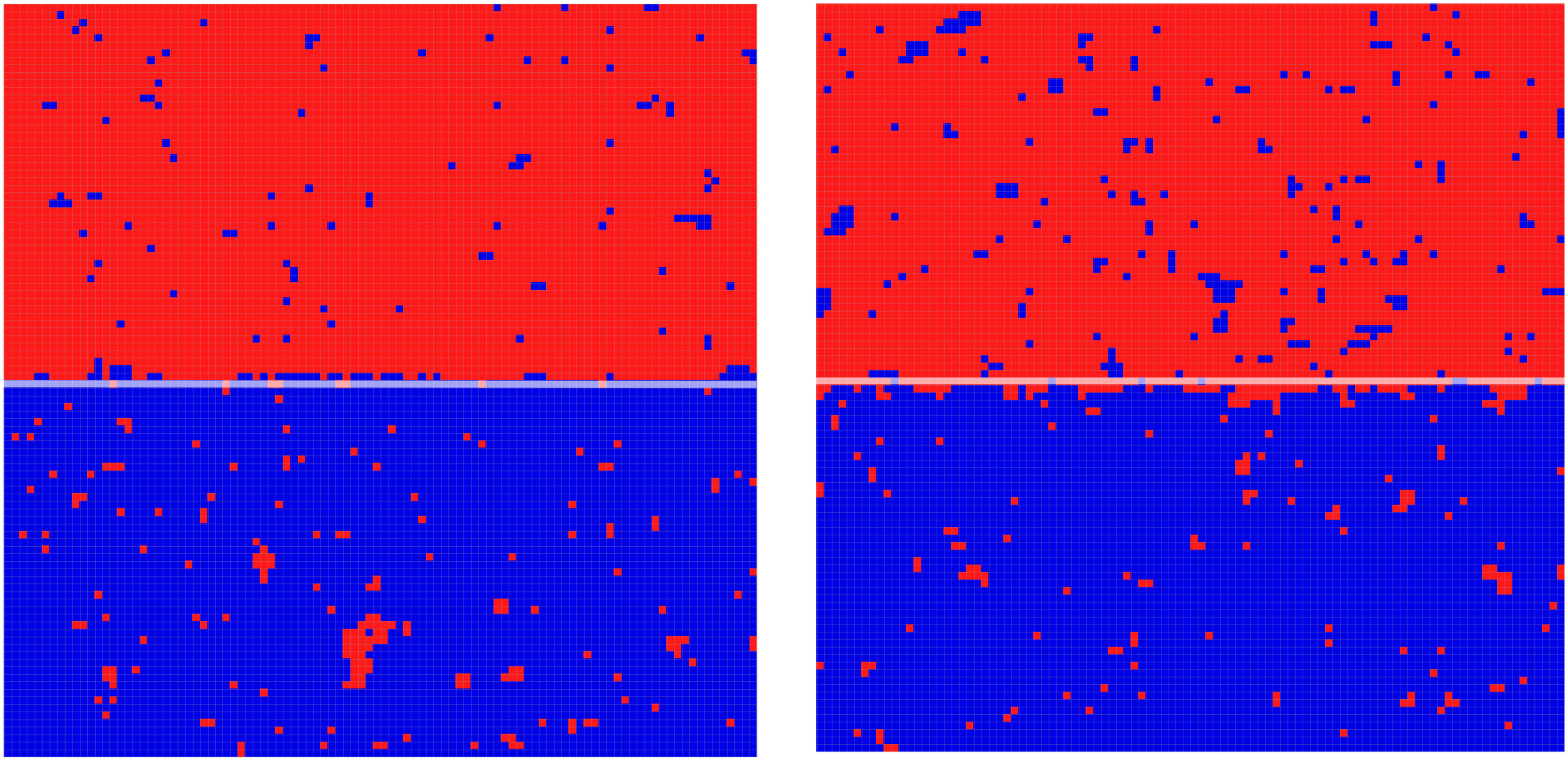
Zoom in



Macroscopic switch

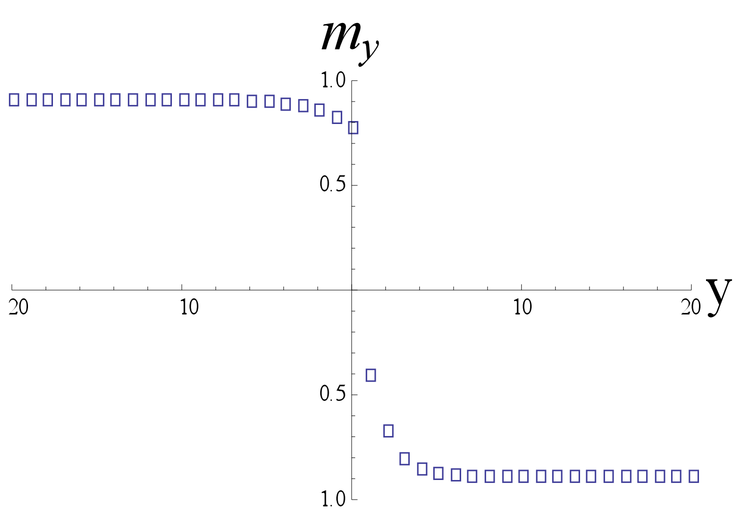
Mesoscopic switches

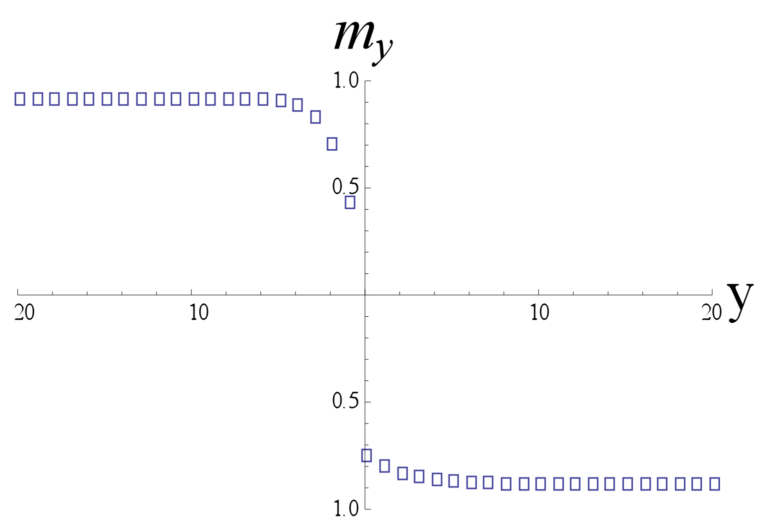
Periodic 30X31 lattice at temperature 0.57Tc.



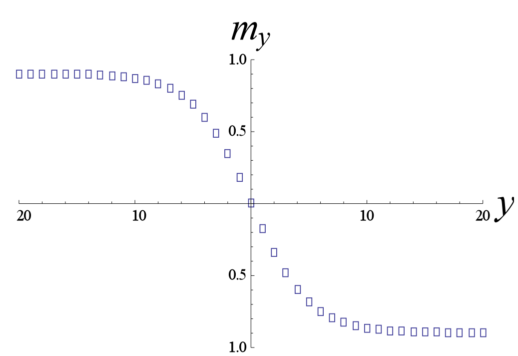
Example of configurations in the two mesoscopic states for a 100X101 with fixed boundary at T=0.85Tc

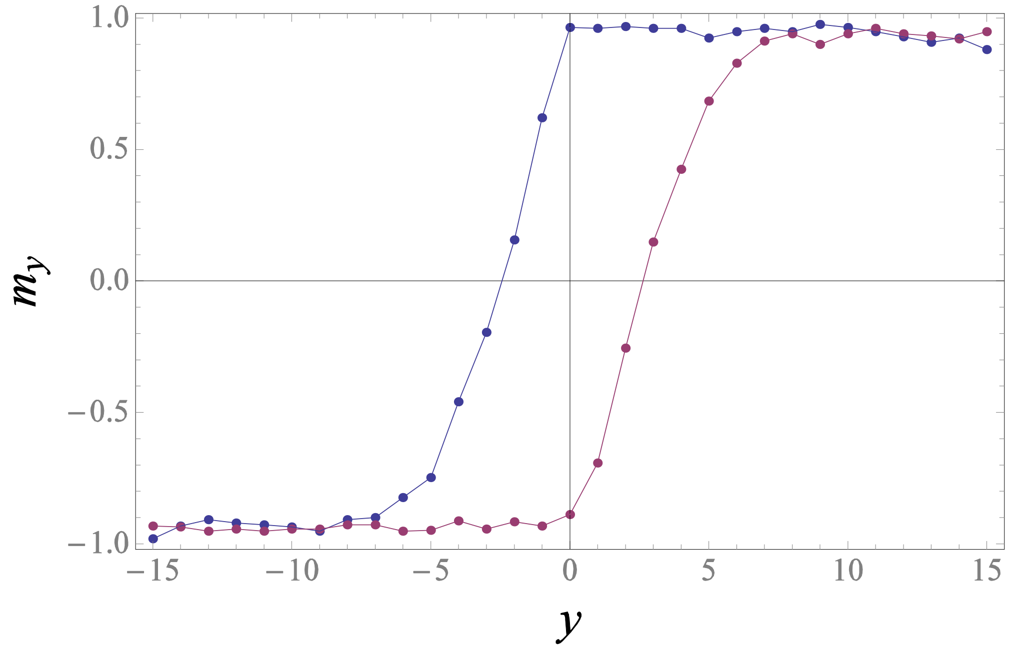
Schematic magnetization profiles



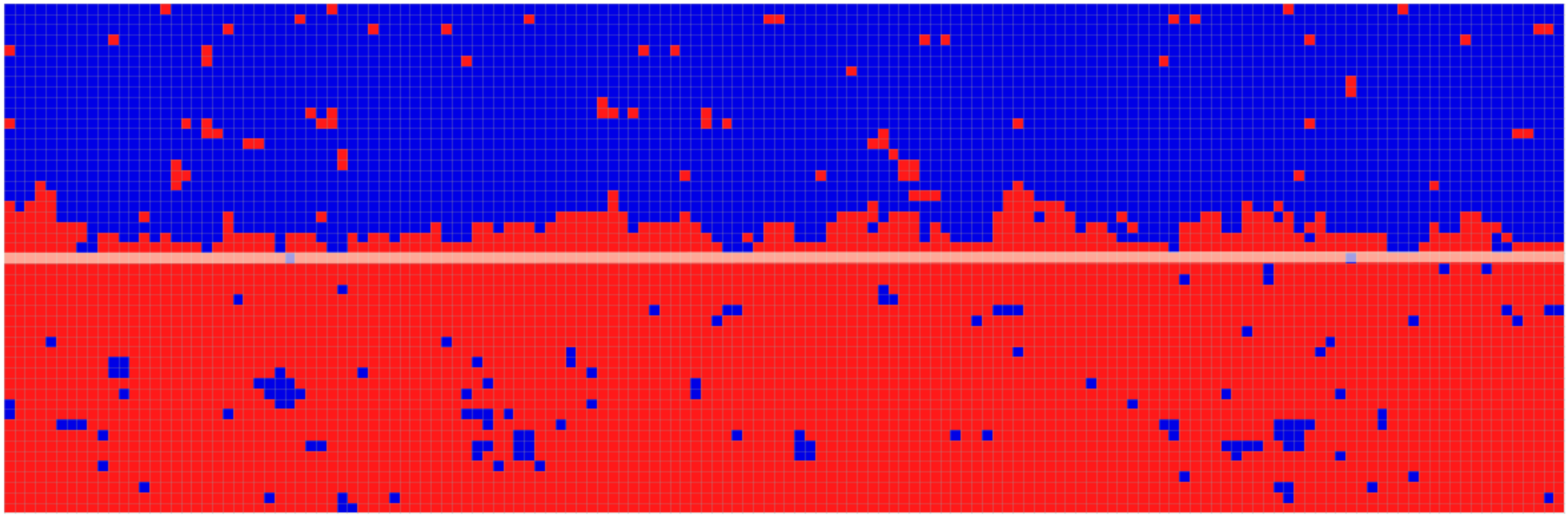


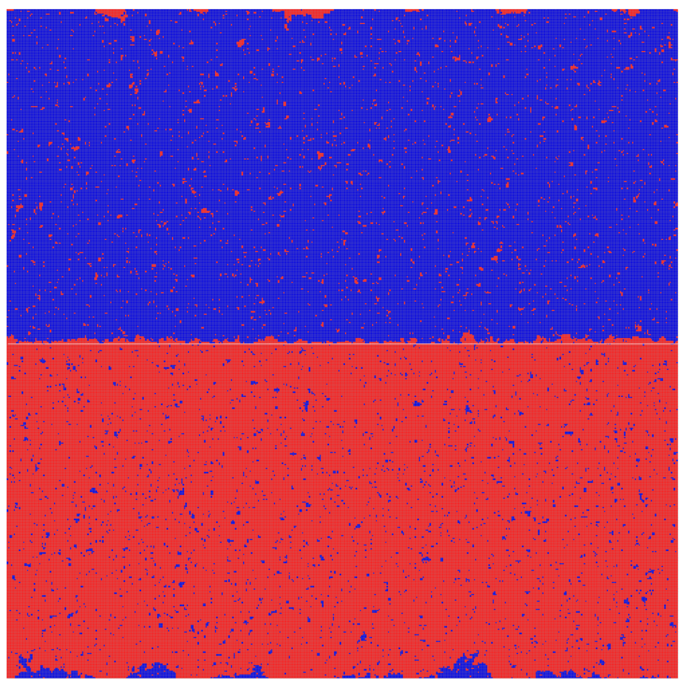
unlike the equilibrium antisymmetric profile



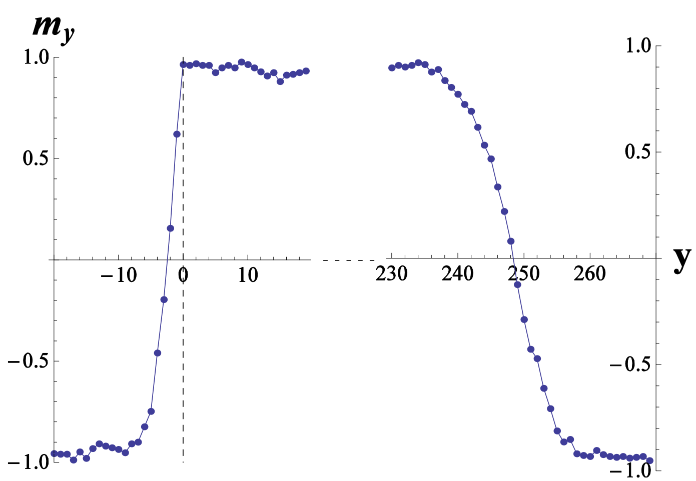


Asymmetric magnetization profile for a periodic 500X501 lattice at temperature T=0.85Tc.





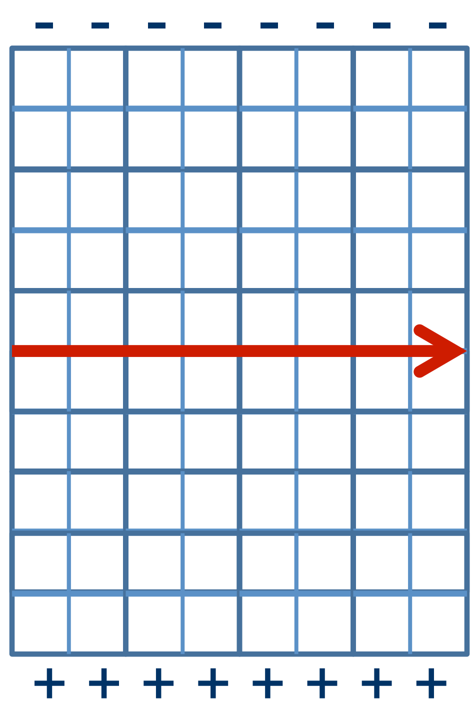
Non-symmetric fluctuations of the interface

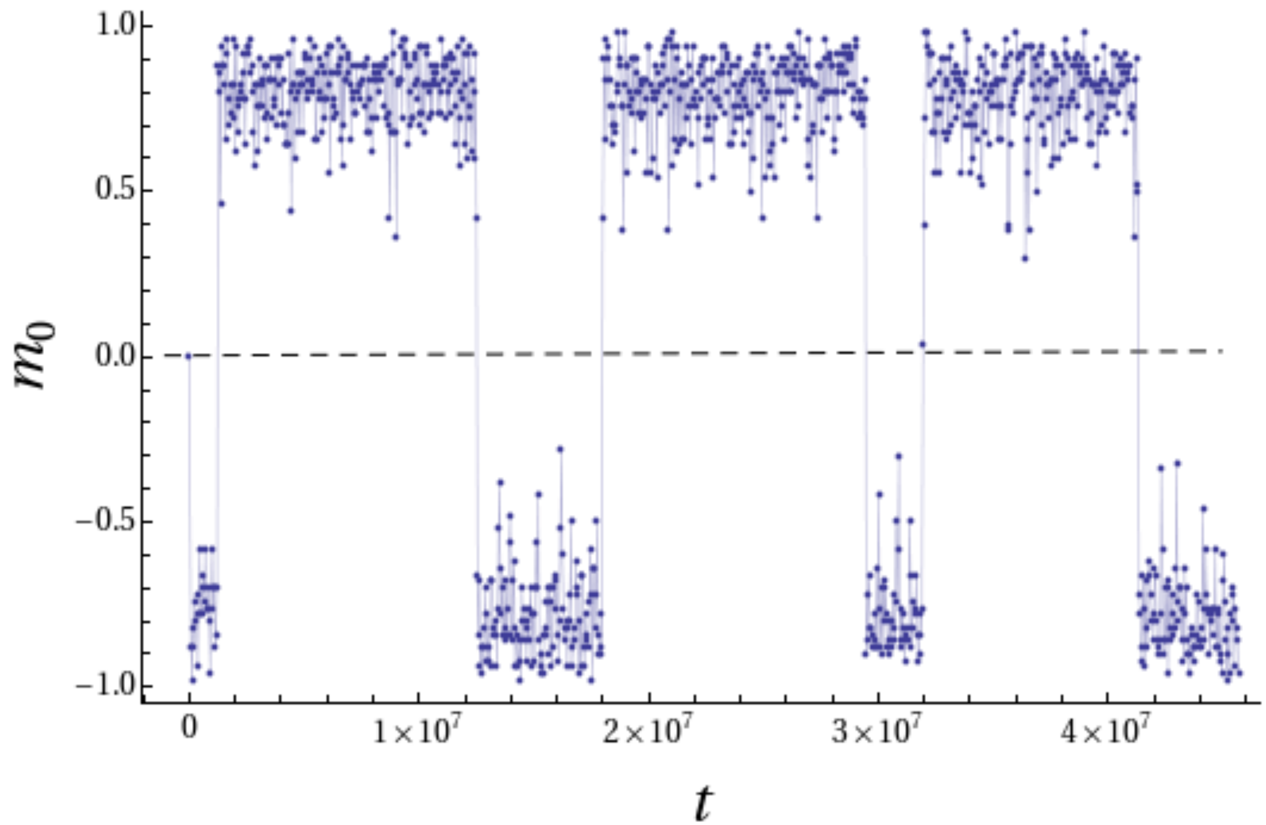


A snapshot of the magnetization profile near the two interfaces on a 500X501 square lattice with periodic boundary condition at T=0.85Tc.

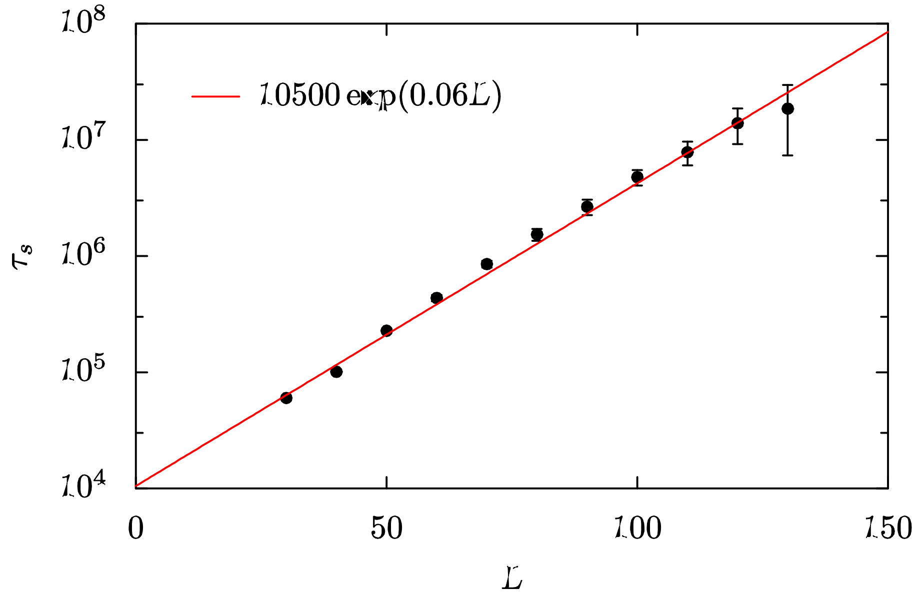
Closed boundary conditions

In order to study the mesoscopic switches in more detail
and to establish the existence of spontaneous symmetry breaking
of the driven line we consider the case of closed boundary conditions



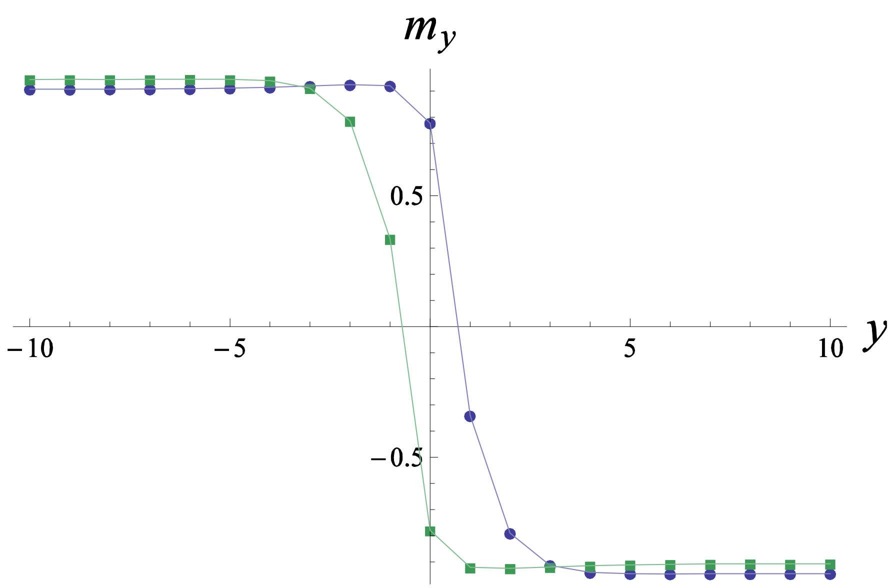


Time series of Magnetization of driven lane for a 100X101 lattice at T= 0.6Tc.



Switching time on a square LX(L+1) lattice with Fixed boundary at T=0.6Tc.

Averaged magnetization profile in the two states



L=100 T=0.85Tc

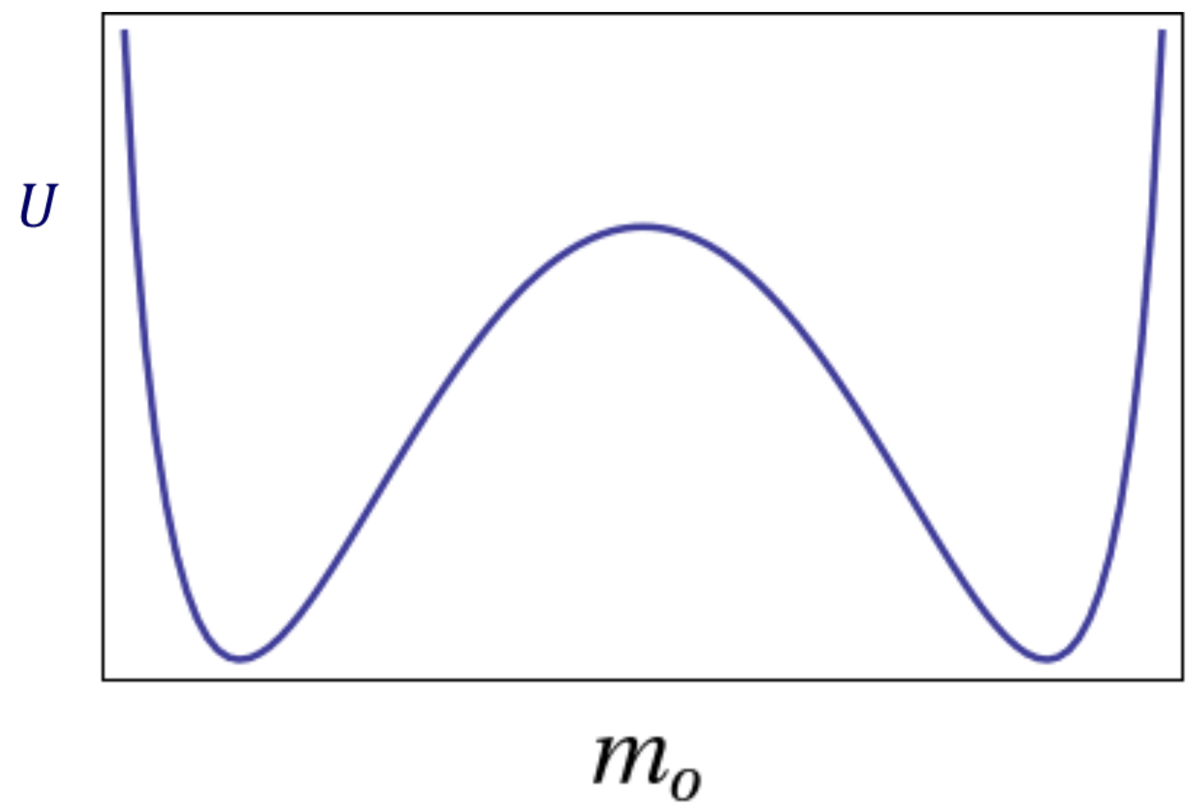
Analytical approach

In general one cannot calculate the steady state measure of this system. However in a certain limit, the steady state distribution (the large deviation function) of the magnetization of the driven line can be calculated.

- Slow exchange rate between the driven line and the rest of the system
- Large driving field $E \gg J$
- Low temperature

In this limit the probability distribution of m_0 is $P(m_0) = e^{-LU(m_0)}$ where the potential (large deviation function) $U(m_0)$ can be computed.

Schematic potential (large deviation function)

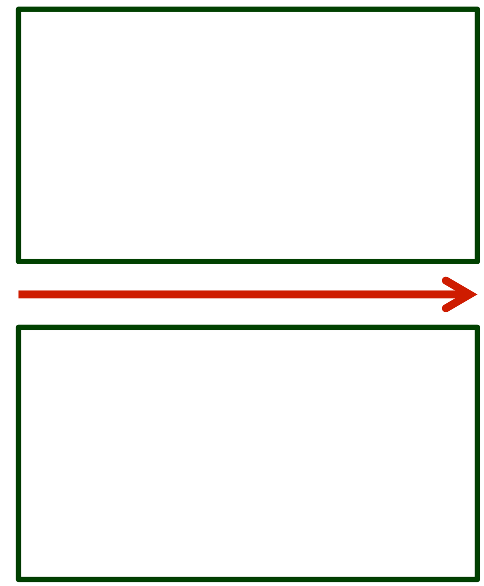


$$P(m_0) = e^{-LU(m_0)}$$

- Slow exchange between the line and the rest of the system

$$+\,- \;\underset{rw(-\Delta H)}{\overset{rw(\Delta H)}{\leftrightarrows}}\; -\,+ \qquad r < O(\frac{1}{L^3})$$

$$w(\Delta H) = \min(1, e^{-\beta \Delta H})$$

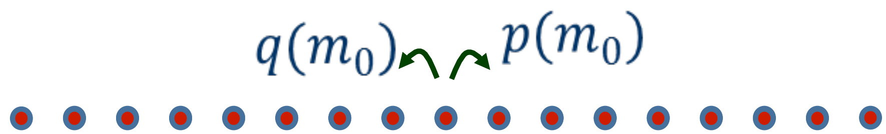


In between exchange processes the systems is composed of 3 sub-systems evolving independently

- Fast drive $E \gg J$
  - the coupling J within the lane can be ignored. As a result the spins on the driven lane become uncorrelated and they are randomly distributed (TASEP)
  - The driven lane applies a boundary field Jm_0 on the two other parts
  - Due to the slow exchange rate with the bulk, the two bulk sub-systems reach the equilibrium distribution of an Ising model with a boundary field Jm_0
- Low temperature limit
  - In this limit the steady state of the bulk sub systems can be expanded in T and the exchange rate with the driven line can be computed.

$$m_o \longrightarrow m_o + \frac{2}{L} \quad \text{with rate} \quad p(m_o)$$

$$m_o \longrightarrow m_o - \frac{2}{L} \quad \text{with rate} \quad q(m_o)$$

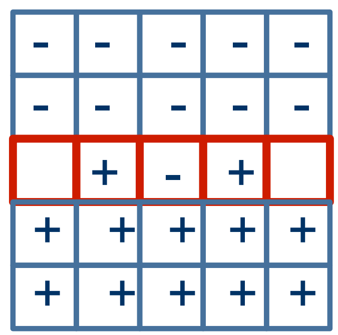


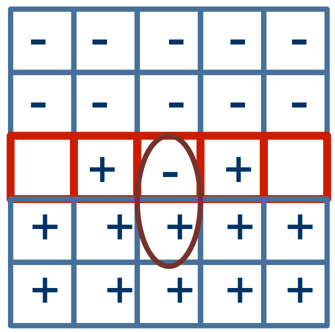
m_o performs a random walk with a rate which depends on m_o

$$P\left(m_o = \frac{2k}{L}\right) = \frac{p(0)\cdots\cdots p(\frac{2(k-1)}{L})}{q(\frac{2}{L})\cdots\cdots q(\frac{2k}{L})} \equiv e^{-U(m_o)}$$

$$U(m_o) = -\sum_{k=0}^{\frac{m_oL}{2}-1} \ln p\left(\frac{2k}{L}\right) + \sum_{k=1}^{\frac{m_oL}{2}} q(\frac{2k}{L})$$

Calculate p(m_o) at low temperature





contribution to p(m_o): $\frac{1}{8}(1-m_o)(1+m_o)^2 e^{-2\beta J} e^{-2\beta J_1}$

J_1 is the exchange rate between the driven line and the adjacent lines

The magnetization of the driven lane $m_\circ$ changes in steps of $2/L$

Expression for rate of increase, p(m_o)

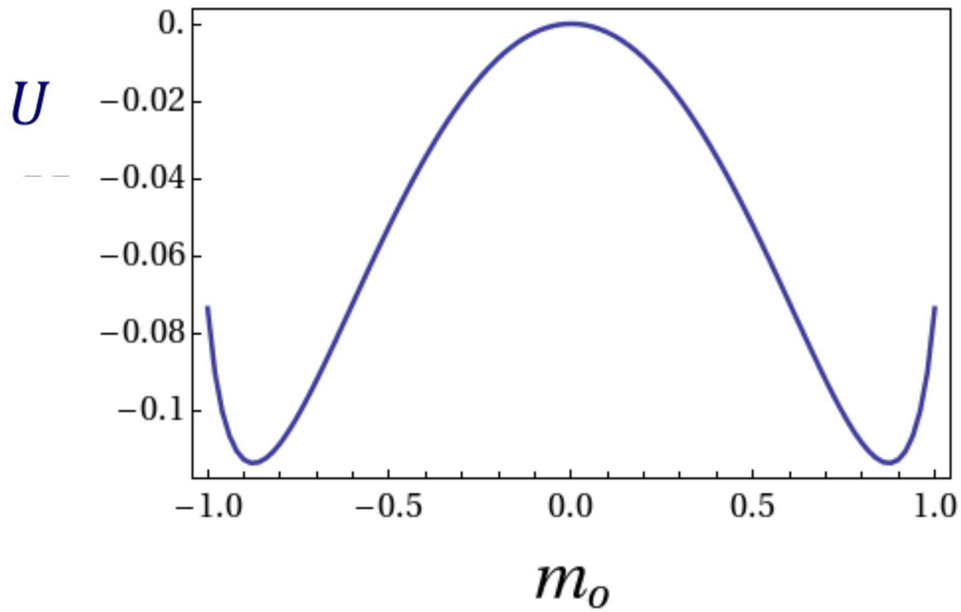
$$p(m_0) = \frac{1}{8}(1 + \mathrm{m}_0)^2(1 - \mathrm{m}_0)e^{-2\beta(J+J_1)}$$

$$+\frac{1}{8}[2(1 + m_0)(1 - m_0)^2\left(e^{-2\beta J_1} + e^{2\beta J_1 m_0}\right)$$

$$+(1 + m_0)^2(1 - m_0)e^{2\beta J_1 m_0} + (1 - m_0)^3 e^{2\beta J_1 m_0}]e^{-6\beta J} + O(e^{-8\beta J})$$

$$q(m_o) = p(-m_o)$$

$$U(m) = -\int_0^m \ln p(k)\, dk + \int_0^m \ln q(k) dk$$



This form of the large deviation function demonstrates the spontaneous symmetry breaking. It also yield the exponential flipping time at finite L. $(T = 0.6T_c, J_1 = J)$

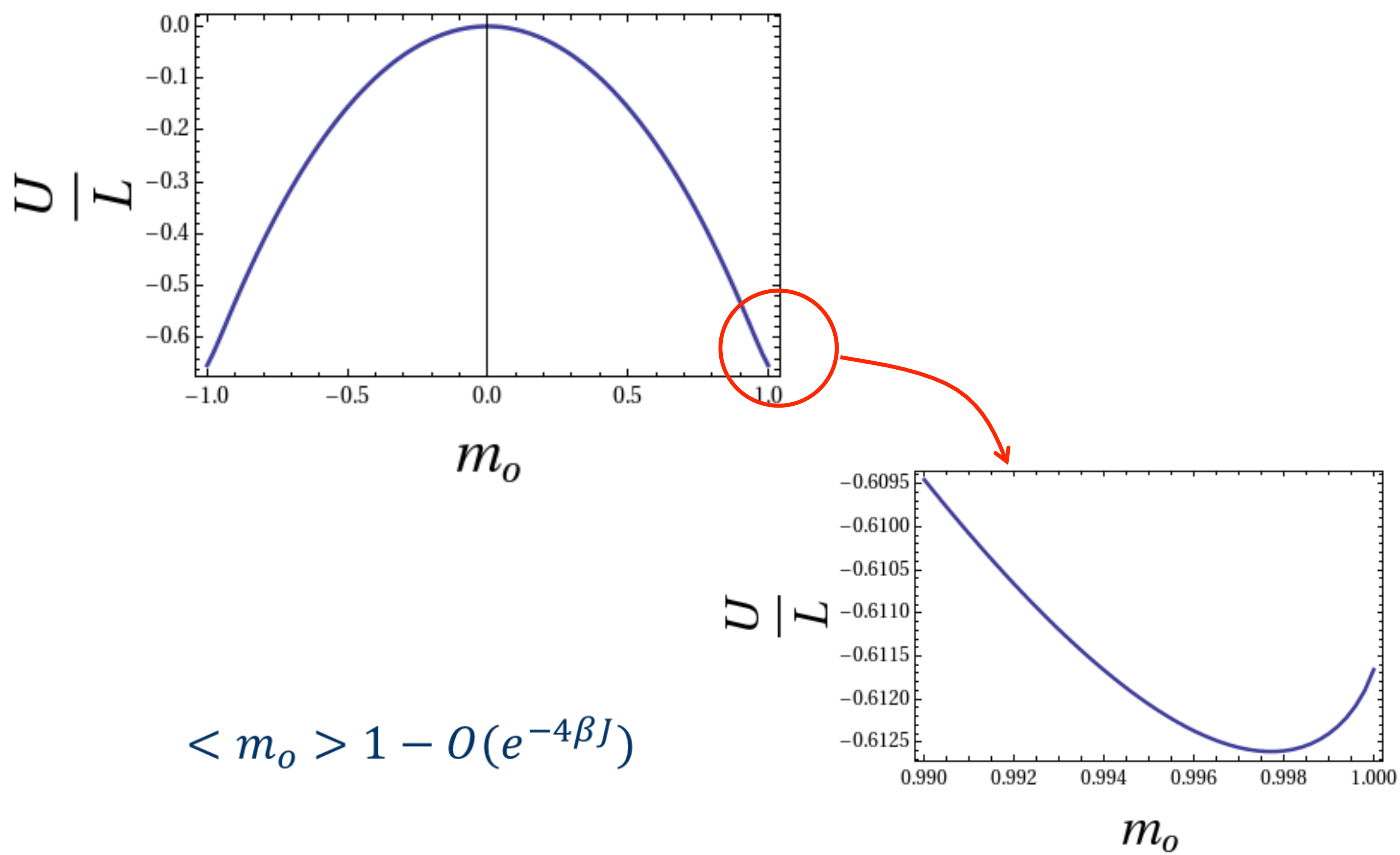
$$P(m_0) = e^{-LU(m_0)}$$

$$< m_o > 1 - O(e^{-4\beta J})$$

Summary

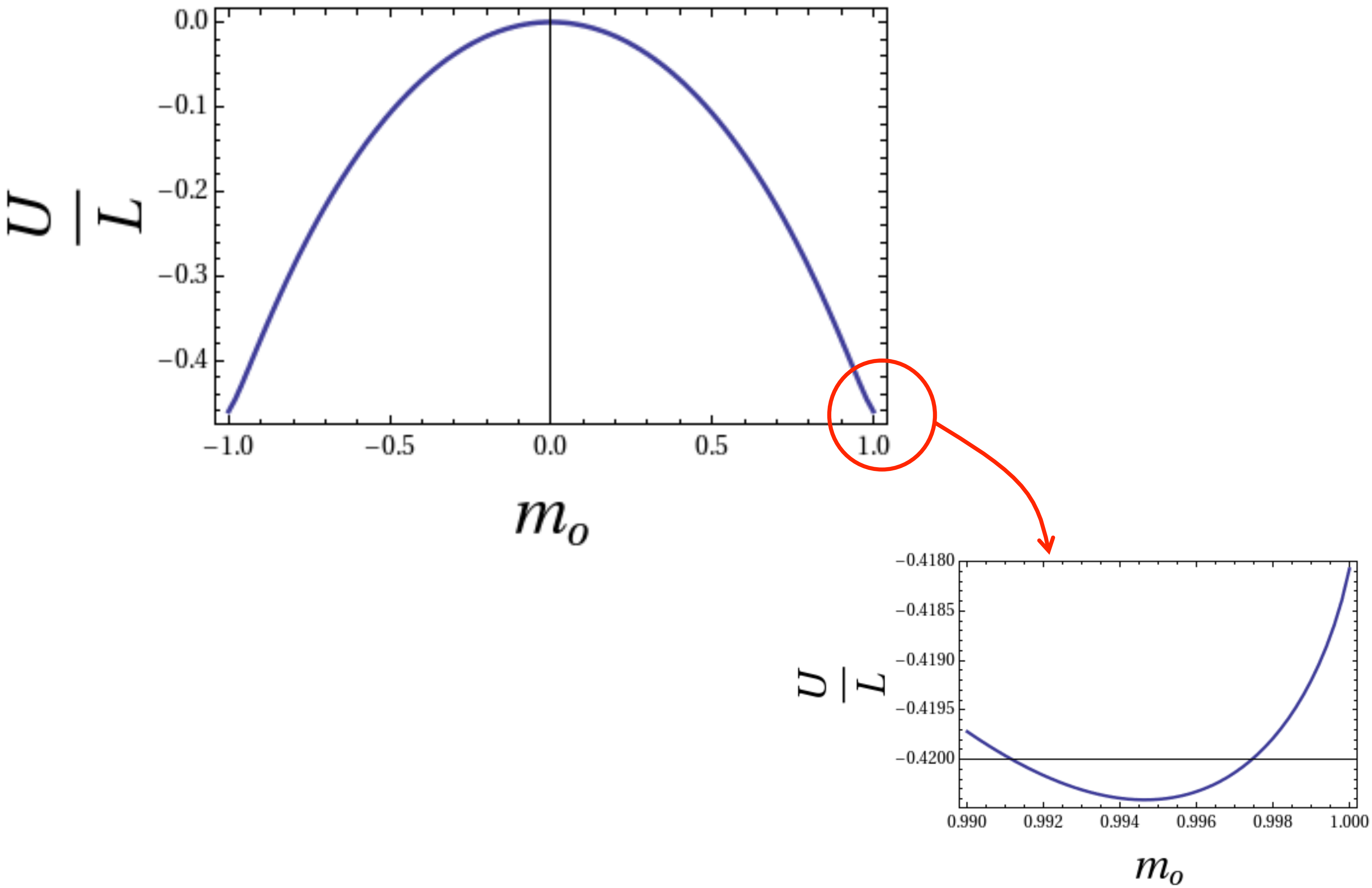
- Driven systems exhibit long range correlations under generic conditions.
- Such correlations sometimes lead to long-range order and spontaneous symmetry breaking which are absent under equilibrium conditions.
- Simple examples of these phenomena have been presented.
- A limit of slow exchange rate is discussed which enables the evaluation of some large deviation functions far from equilibrium.

$$\beta J = \frac{3}{2}\,, \qquad \beta J_1 = \frac{J}{5}$$



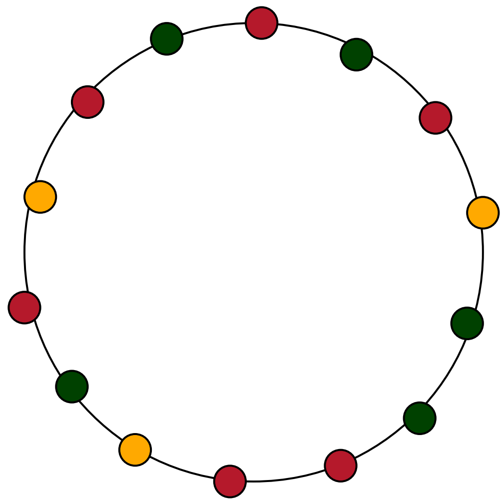
$$< m_o > 1 - O(e^{-4\beta J})$$

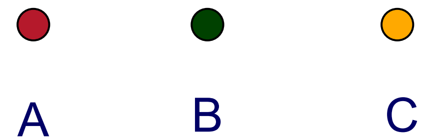
J=3/2 J1=0



0.0
−0.1
−0.2
−0.3
−0.4
−1.0
−0.5
0.0
0.5
1.0
$\frac{U}{L}$
m_o
−0.4180
−0.4185
−0.4190
−0.4195
−0.4200
0.990
0.992
0.994
0.996
0.998
1.000
$\frac{U}{L}$
m_o

Example III: ABC Model- phase separation in d=1





dynamics

$$AB \underset{1}{\overset{q}{\rightleftarrows}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftarrows}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftarrows}} AC$$

q=1 corresponds to equilibrium and the steady state is homogeneous (fully mixed).
question: what is the steady state for q≠1?

Evans,Kafri, Koduvely, Mukamel PRL 80, 425 (1998)

Simple argument:

ACCCC → CCCCA

CBBBB → BBBBC

BAAAA → AAAAB

$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$

$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$

$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$

…AACBBBCCAAACBBBCCC…

↓ fast rearrangement

…AABBBCCCAAABBBCCCC…

↓ slow coarsening

…AAAAABBBBBCCCCCCAA…

The model reaches a phase separated steady state

The model exhibits strong phase separation

…AAAAAAAABBBABBBBBBCCCCCCCCCCAA…

The probability of a particle to be at a distance l on the wrong side of the boundary is q^l

The width of the boundary layer is $-1/\ln q$

Special case $N_A = N_B = N_C$

The argument presented before is general, independent of densities.

For the equal densities case the model has detailed balance for arbitrary q.

We will demonstrate that for any microscopic configuration {X} one can define "energy" E({X}) such that the steady state distribution is

$$P(\{X\}) \propto q^{E(\{X\})} \propto e^{\ln q E(\{X\})}$$

AAAAAABBBBBBCCCCCC E=0

……AB….. $\longrightarrow$ ……BA….. E $\rightarrow$ E+1

……BC….. $\longrightarrow$ ……CB….. E $\rightarrow$ E+1

……CA….. $\longrightarrow$ ……AC….. E $\rightarrow$ E+1

With this weight one has:

$$\underbrace{W(AB \rightarrow BA)}_{=q}P(\dots AB\dots) = \underbrace{W(BA \rightarrow AB)}_{=1}P(\dots BA\dots)$$

$$P(\dots BA\dots)/P(\dots AB\dots) = q$$

This definition of “energy” is possible only for $N_A = N_B = N_C$

AAAAABBBBBCCCCC → AAAABBBBBCCCCC**A**

$E \rightarrow E+N_B-N_C$

$N_B = N_C$

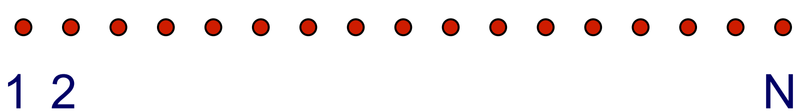
Thus such “energy” can be defined only for $N_A=N_B=N_C$

$$N_A = N_B = N_C$$

$$P(\{x\}) = q^{E(\{x\})}$$

The "energy" E may be written as

$$E(\{x\}) = \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - (N/3)^2$$



(long-range interaction)

Alternatively, in a manifestly translational invariant form:

$$E(\{x\}) = \sum_{i=1}^{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N} \right) \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right)$$

$$P(\{x\}) = q^{E(\{x\})}$$

$$E(\{x\}) = \sum_{i=1}^{N-1} \sum_{k=1}^{N-i} \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right) - (N/3)^2$$

$$E(\{x\}) = \sum_{i=1}^{N} \sum_{k=1}^{N-1} \left(1 - \frac{k}{N} \right) \left(C_i B_{i+k} + A_i C_{i+k} + B_i A_{i+k} \right)$$

- Local dynamics
- Long range interaction

$$AB \underset{1}{\overset{q}{\rightleftharpoons}} BA$$

$$BC \underset{1}{\overset{q}{\rightleftharpoons}} CB$$

$$CA \underset{1}{\overset{q}{\rightleftharpoons}} AC$$

Summary

- Driven systems exhibit long range correlations under generic conditions.

- Such correlations sometimes lead to long-range order and spontaneous symmetry breaking which are absent under equilibrium conditions.

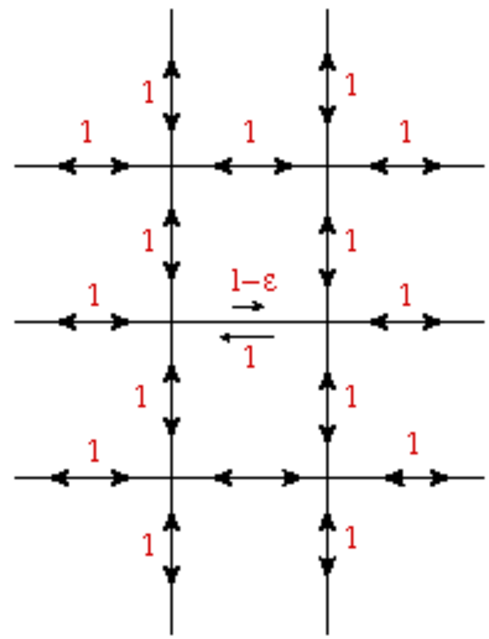
- Simple examples of these phenomena have been presented.

- A limit of slow exchange rate is discussed which enables the evaluation of some large deviation functions far from equilibrium.

$$(V-1)P(k)+P(0)=N$$

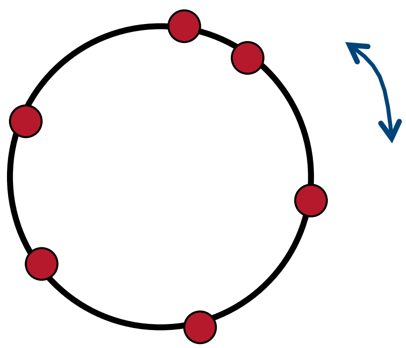
$$P(k)=\frac{N-P(0)}{V-1}\approx\frac{N}{V}+O(\frac{1}{V})$$

$$p(k)=\begin{cases}\dfrac{Ne^{-\beta u}}{V-1+e^{-\beta u}}\approx\dfrac{N}{V}e^{-\beta u} & k=0\\[2ex]\dfrac{N}{V-1+e^{-\beta u}}\approx\dfrac{N}{V} & k\neq 0\end{cases}$$

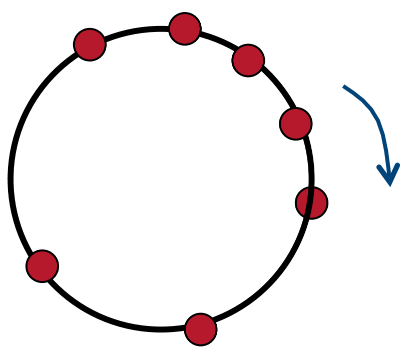


1
1
1
1
1
1
1
1−ε
1
1
1
1
1
1
1
1
1

Driven systems typically exhibit long-range correlations in their steady states.

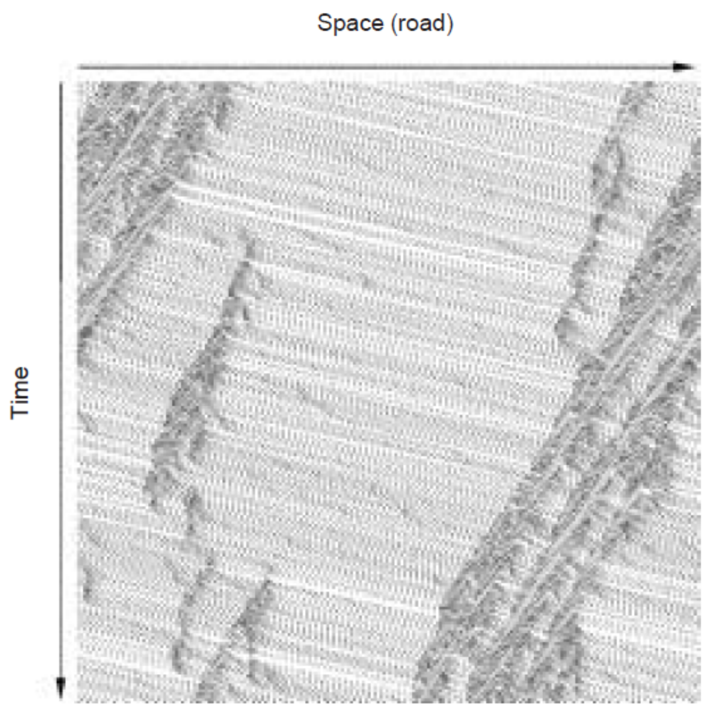


In equilibrium - no phase separation
(the density is macroscopically homogeneous)



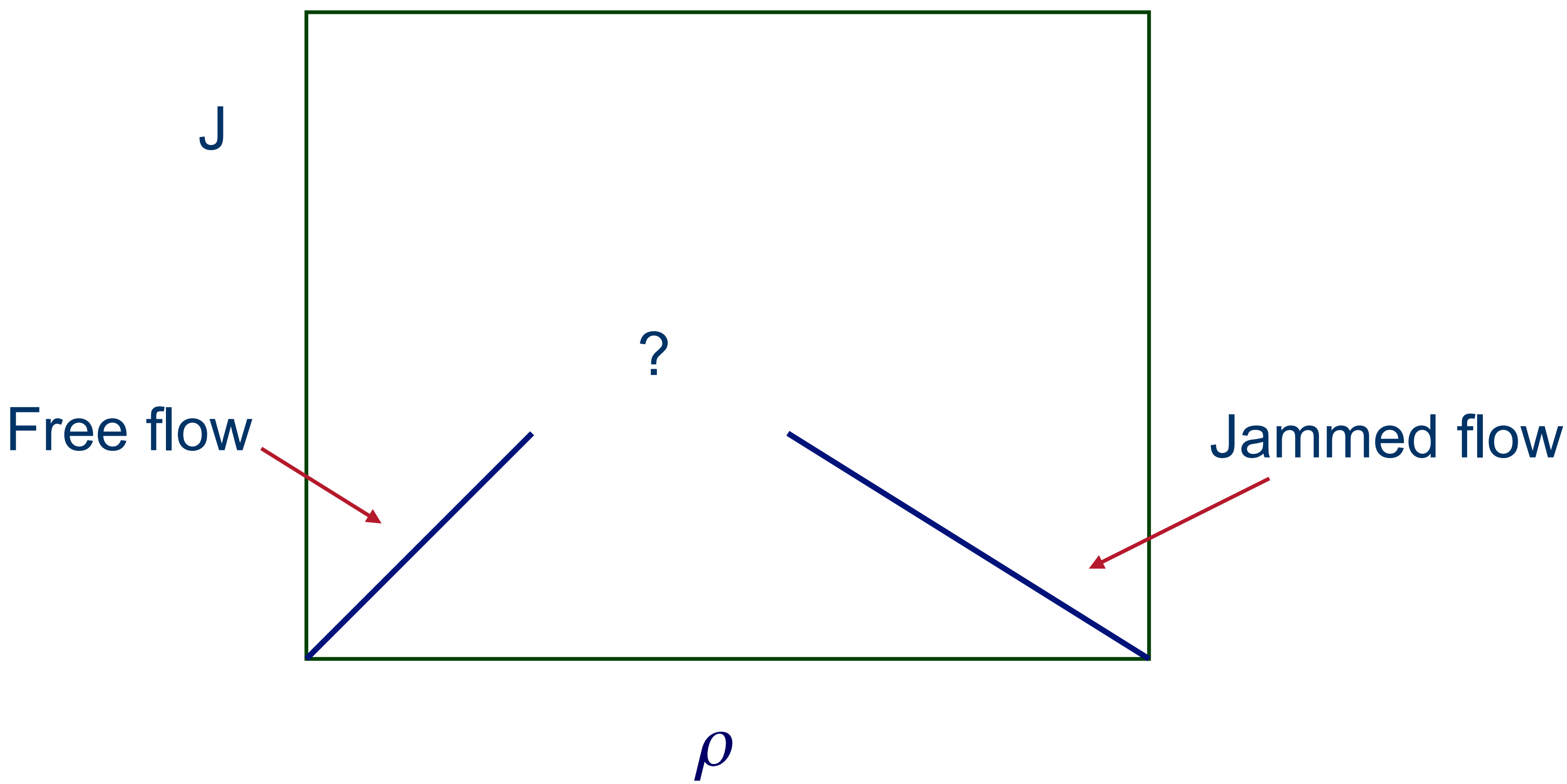
In driven systems – phase separation can take place
("liquid-gas" transition in one dimension)

Traffic flow

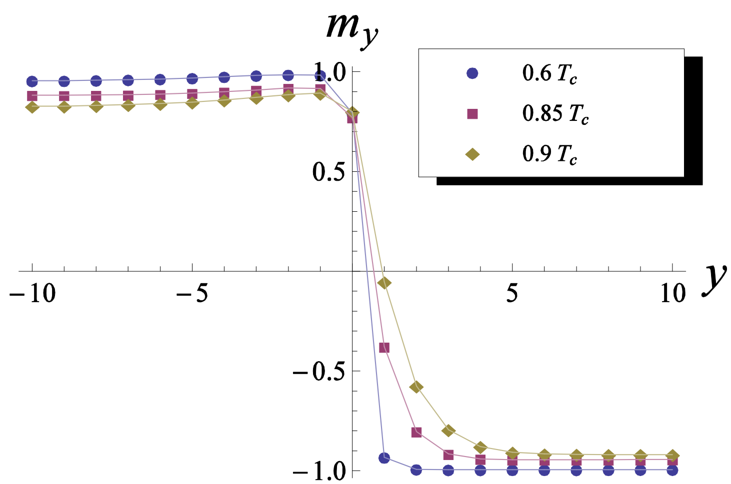


Single-lane traffic model

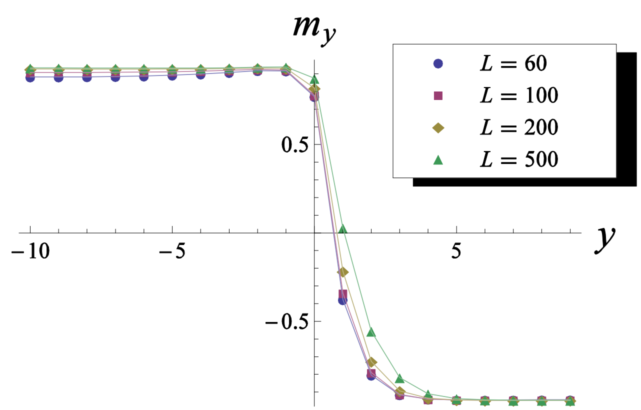
Fundamental Diagram



Is there a jamming phase transition?
or is it a broad crossover?



Variation with temperature. Fixed boundary, 60X61 lattice.



Variation with systems size. Fixed boundary, T=0.85Tc.

+ - → - +   with rate $\min(1, e^{-\beta\Delta H+\beta E})$

- + → + -   with rate $\min(1, e^{-\beta\Delta H -\beta E})$

A snapshot of the magnetization profile in the two states