A large deviation approach to computing rare transitions in multi-stable stochastic turbulent flows

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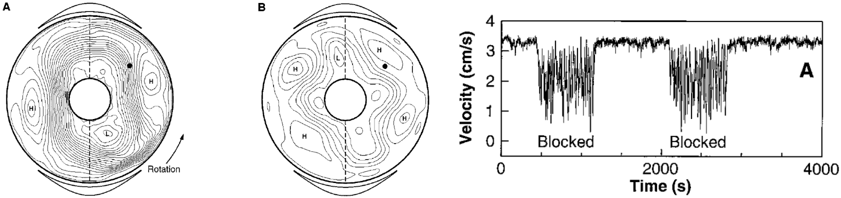
Bistability in Rotating Tank Experiment

Transitions between blocked and zonal states



Weeks, Tian, Urbach, Ide, Swinney, Ghil, Science, 1997

- Strong analogy to weather regimes in the Earth's atmosphere

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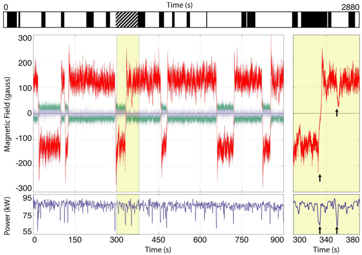
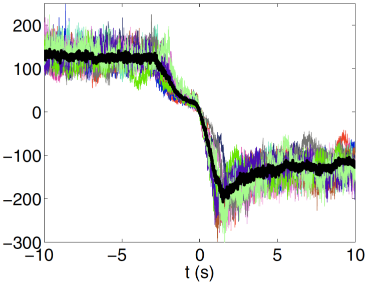
Bistability in the VKS Experiment

Transitions in the polarization of the magnetic field





Berhanu *et al.* EPL, 2007

- Transition trajectories may be concentrated around a single trajectory

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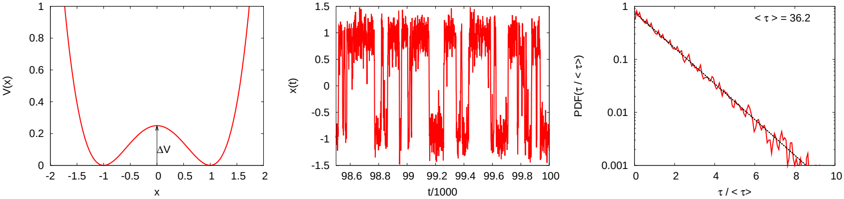
Classical Bistability: Double-Well Potential

$$\dot{x}(t) = -\frac{dV}{dx} + \sqrt{k_B T}\eta(t)$$



- Gradient system with a known energy landscape
- Arrhenius law for transition rate: $k = A \exp\left(-\frac{\Delta V}{k_B T}\right)$ Arrhenius 1889
- Turbulent flows do not fall into this framework
- Modern approaches include Freidlin–Wentzell theory (mathematics) and path integrals and instantons (physics)

Aim of this Talk

- Large deviation of bistability in turbulent flows
- We study the 2D stochastic Navier-Stokes equations (simplest turbulence model)
- Computation of instantons with a minimum action method

Differences to classical bistability phenomenon

- Non-gradient dynamics, connected steady states, unknown steady states, complexity issues
- Diffusion across steady states may prevent rare transitions, bistability and large deviation results

The 2D Stochastic Navier-Stokes Equations

$$\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega = \underbrace{-\alpha\omega + \nu\Delta\omega}_{\textit{Dissipation}} \underbrace{+\sqrt{2\alpha}\eta}_{\textit{Forcing}}$$

$$\omega = (\nabla \times \mathbf{v}) \cdot \mathbf{e}_z, \qquad \mathbf{v} = \mathbf{e}_z \times \nabla\psi, \qquad \omega = \Delta\psi$$

- Stochastic white in time forcing:

$$\langle \eta(\mathbf{x}, t)\eta(\mathbf{x}', t') \rangle = C(\mathbf{x} - \mathbf{x}')\delta(t - t')$$

- Doubly periodic domain $\mathcal{D}$
- Consider the weak forcing and dissipation regime: $\nu \ll \alpha \ll 1$
- Timescale separation: $\tau_{energy} = 1 \ll 1/\alpha = \tau_{dissipation}$

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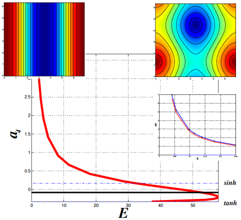
Leading Order Dynamics – The 2D Euler Equations

$$
\begin{aligned}
\frac{\partial \omega}{\partial t} + \mathbf{v} \cdot \nabla \omega &= 0 \\
\omega = (\nabla \times \mathbf{v}) \cdot \mathbf{e}_z, \quad \mathbf{v} &= \mathbf{e}_z \times \nabla \psi, \qquad \omega = \Delta \psi
\end{aligned}
$$

- The 2D Euler equations have an infinite number of steady states:
  $\mathbf{v} \cdot \nabla \omega = 0 \Rightarrow \omega = f(\psi)$
- The flow self-organizes and converges toward steady states (attractors)



- Robert–Miller–Sommeria equilibrium statistical mechanics predicts which states can be observed (what $f(\cdot)$ is selected)

Bouchet and Venaille, Physics Reports, 2012

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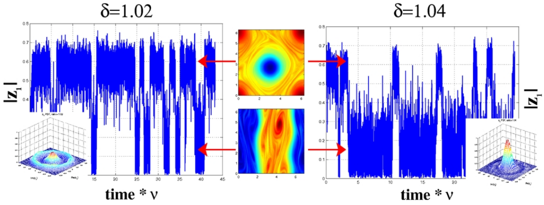
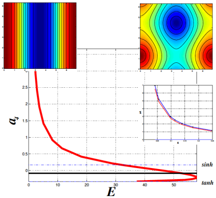
Bistability in the 2D Stochastic Navier-Stokes Equations

Transitions between dipole and parallel flow states

- $z_1 = \int_{\mathcal{D}} \omega(\mathbf{x},t)\exp(iy)\,\mathrm{d}\mathbf{x}$





Bouchet and Simonnet, PRL, 2009

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The Onsager–Machlup Path Integral

The transition probability

Consider a transition from state ω_0 to state ω_T in time T:

$$P(\omega_0, 0; \omega_T, T) = \int \mathcal{D}[\omega] e^{-\frac{1}{2\alpha}\mathcal{A}(\omega)}$$

The action functional

$$\begin{aligned}
\mathcal{A}(\omega) &= \frac{1}{2}\int_0^T \int_{\mathscr{D}} p(\mathbf{x}, t) C^{-1}(\mathbf{x} - \mathbf{x}') p(\mathbf{x}', t) \, \mathrm{d}\mathbf{x} \, \mathrm{d}\mathbf{x}' \, \mathrm{d}t \\
p &= \dot{\omega} + \mathbf{v} \cdot \nabla\omega + \alpha\omega - \nu\Delta\omega
\end{aligned}$$

- Any deterministic trajectory ($p = 0$) has zero action: $\mathcal{A} = 0$

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The Saddle-Point Approximation ($\alpha \ll 1$)

Which trajectory maximizes the transition probability P?

$$P(\omega_0, 0; \omega_T, T) = \int \mathcal{D}[\omega] e^{-\frac{1}{2\alpha}\mathcal{A}(\omega)}$$

- The most probable transition trajectory minimizes $\mathcal{A}(\omega)$:

$$\omega^* = \underset{\{\omega | \omega(0)=\omega_0,\, \omega(T)=\omega_T\}}{\arg\min} \mathcal{A}(\omega)$$ The Instanton Trajectory

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Large Deviation Principle (same as Freidlin–Wentzell)

$$\lim_{\alpha \to 0} -\alpha \ln(P) = \mathcal{A}(\omega^*)$$

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Exact Results: Large Deviations for Rare States

We can explicitly compute instantons for particular cases:

- White in space forcing: $C(\mathbf{x}-\mathbf{x}') = \delta(\mathbf{x}-\mathbf{x}')$
- Parallel flows (flows with symmetry)
- States that are eigenmodes of the Laplacian

For the white noise case, we have the following large deviation result:

$$P_s(\omega) \underset{\mathcal{Z}\to\infty}{\simeq} e^{-\frac{1}{2}\int_{\mathcal{D}} \omega^2 \, \mathrm{d}\mathbf{x}}$$

where $\mathcal{Z} = \frac{1}{2}\int_{\mathcal{D}} \omega^2 \, \mathrm{d}\mathbf{x}$ is the enstrophy and $P_s = \lim_{T\to\infty} P$

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Non-Isolated Steady States Lead to Non-Standard Large Deviations

Attractors of the 2D Euler equations (equilibrium)

- The 2D Euler equations contain non-isolated attractors
- Any steady state ω is connected to zero through a continuous path of steady states: $s\omega(st), 0 \leq s \leq 1$
- Therefore, any two steady states, ω_1 and ω_2 can be connected through a continuous path of steady states (attractors are non-isolated)

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2D Navier-Stokes equations (non-equilibrium)

- Dynamics can slowly diffuse across steady states: $\tau \sim 1/\alpha$
- For transitions between steady states: $\mathcal{A}(\omega^*) \to 0$ as $\alpha \to 0$

Transition is not rare!

No large deviation and no bistability

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The Importance of Degenerate Forcing

Strategy: If we can prevent diffusion across steady states, then transitions between two steady states will become a rare event

Force Correlation: $\langle \eta(\mathbf{x},t)\eta(\mathbf{x}',t') \rangle = C(\mathbf{x}-\mathbf{x}')\delta(t-t')$

- Definition: $C_{\mathbf{k}} = \int_{\mathcal{D}} C(\mathbf{x}) \exp(i\mathbf{k}\cdot\mathbf{x}) \mathrm{d}\mathbf{x}$, if $C_{\mathbf{k}} = 0$ for some $\mathbf{k}$, the force is called degenerate, otherwise non-degenerate
- If the forcing is non-degenerate, the dynamics can diffuse across continuous sets of steady states ($\mathcal{A} \to 0$)

  Then there is no large deviation and no bistability
- What about if we set $C_{\mathbf{k}} = 0$ at the largest scales (the scale of the attractors)?

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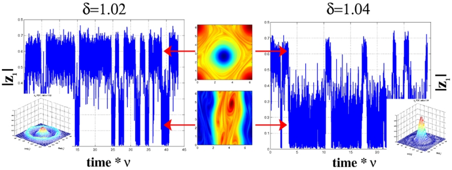
- What about if we set $C_{\mathbf{k}} = 0$ at the largest scales (the scale of the attractors)?

  The transition at the largest scale will have to be excited via nonlinear interactions

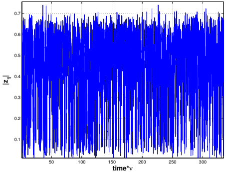
Bistability with Degenerate Forcing



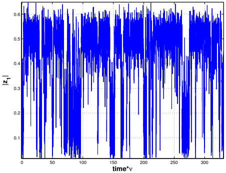
Bouchet and Simonnet, PRL, 2009

- $z_1 = \int_{\mathcal{D}} \omega(\mathbf{x}, t) \exp(iy) \, \mathrm{d}\mathbf{x}$
- Bistability becomes more apparent as forcing becomes more degenerate

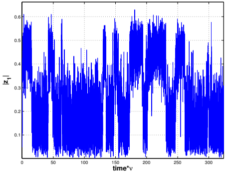
Increasing Degeneracy



$2 \le |\mathbf{k}| \le 7$



$3 \le |\mathbf{k}| \le 7$



$4 \le |\mathbf{k}| \le 7$

Numerical Computation of Instantons

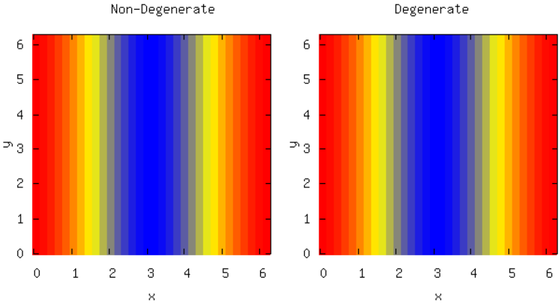
- We implement a variational approach to determine the instanton trajectory by minimizing $\mathcal{A}(\omega)$ (minimum action method) E, Ren, Vanden-Eijnden, 2004
- The initial and final states are fixed throughout the minimization
- We iteratively minimize an initial guess, simultaneously over space and time, in a descent direction d_n:

$$\omega_{n+1} = \omega_n + l_n d_n$$

- Newton or quasi-Newton methods (BFGS) are too expensive to implement
- We utilize a nonlinear conjugate gradient method with central differencing scheme in time and pseudo-spectral in space

Numerical Instantons: Non-Degenerate vs. Degenerate

Transition between a parallel flow and dipole



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Conclusions

- The 2D stochastic Navier-Stokes equations are a non-gradient system with non-isolated steady states
- Because the set of attractors are connected, the classical phenomenology may not hold
- Feasible to numerically compute instantons using a minimum action method
- No bistability for non-degenerate forcing
- We have explicit large deviation predictions for rare stationary probabilities

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