Large deviations of the density and of the current in non equilibrium steady states

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Outline

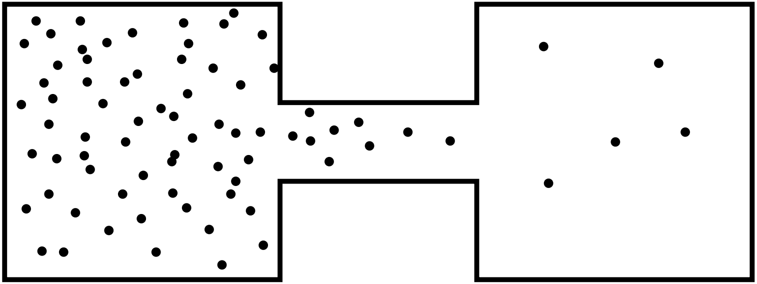
Density fluctuations in non-equilibrium systems

Current fluctuations

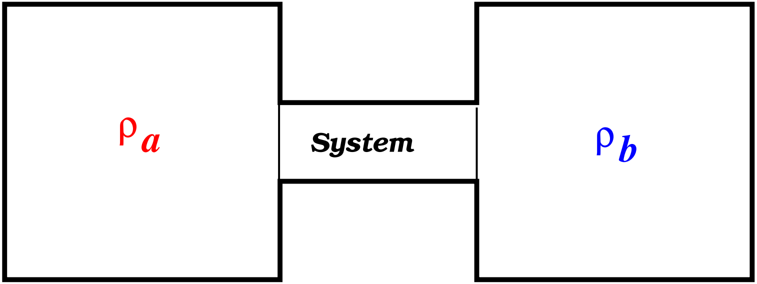
Open questions

- Non steady state situations
- Deterministic dynamics

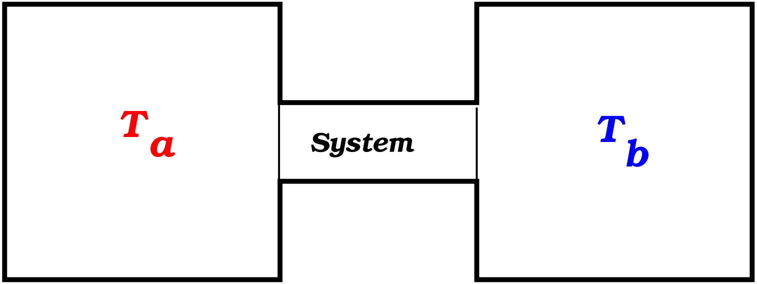
Non equilibrium steady states: current of particles

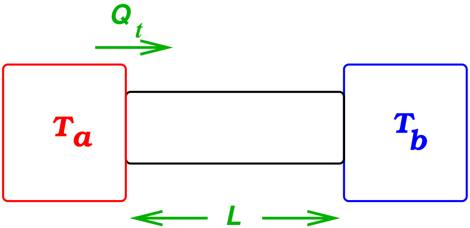


Non equilibrium steady states: current of particles



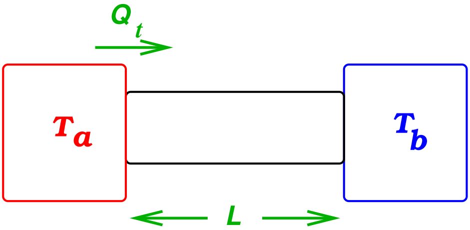
Non equilibrium steady states: current of heat





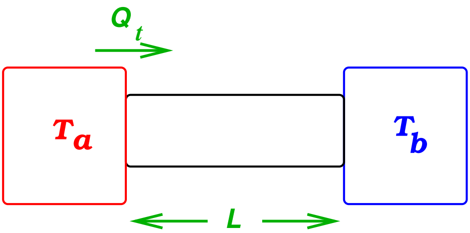
Equilibrium ($T_a = T_b$)

Steady state ($T_a \neq T_b$)



Equilibrium ($T_a = T_b$) Steady state ($T_a \neq T_b$)

$$P(Q) = P(-Q)$$



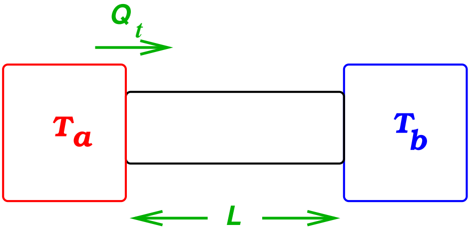
Equilibrium ($T_a = T_b$)

$$P(Q) = P(-Q)$$

Steady state ($T_a \neq T_b$)

$$P(Q) \sim P(-Q) \exp\left[Q\left(\frac{1}{kT_b} - \frac{1}{kT_a}\right)\right]$$

Fluctuation Theorem



Equilibrium ($T_a = T_b$)

$$P(Q) = P(-Q)$$

$$\frac{\langle Q^2 \rangle}{t} \sim \frac{1}{L}$$

Steady state ($T_a \neq T_b$)

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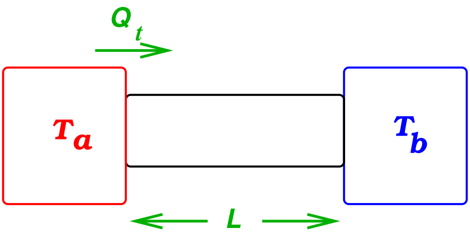
Fluctuation Theorem

$$\frac{\langle Q \rangle}{t} \sim \frac{A(T_a, T_b)}{L}$$

Fourier's law



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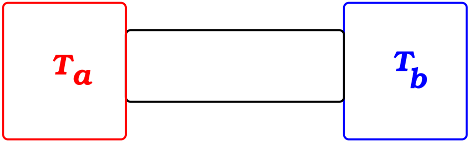
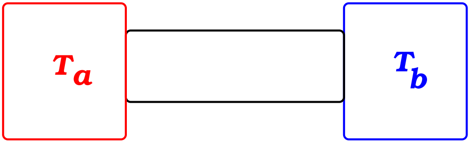
Fourier's law

$$P(Q)?$$



Equilibrium ($T_a = T_b$)

Non-equilibrium steady state ($T_a \neq T_b$)



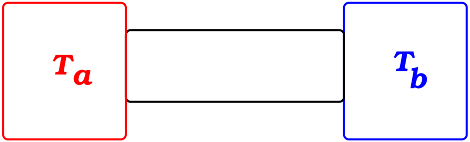
Equilibrium $(T_a = T_b)$

Non-equilibrium steady state $(T_a \neq T_b)$

$$P(\mathcal{C}) \sim \exp\left[-\frac{E(\mathcal{C})}{kT}\right]$$

$$\boxed{P(\mathcal{C}) = ?}$$

Density fluctuations



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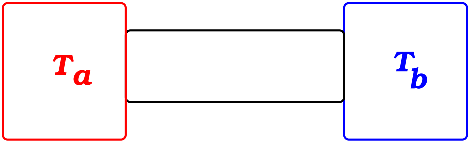
Density fluctuations

Short range correlations

Long range correlations



Equilibrium $(T_a = T_b)$

Non-equilibrium steady state $(T_a \neq T_b)$

$$P(\mathcal{C}) \sim \exp\left[-\frac{E(\mathcal{C})}{kT}\right]$$

$$\boxed{P(\mathcal{C}) = ?}$$

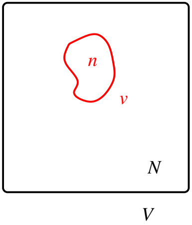
Density fluctuations

Short range correlations
Time symmetry of fluctuations

Long range correlations
Time asymmetry of fluctuations

Density fluctuations at equilibrium

Starting from $S = k \log \Omega$ Einstein



N *particles*

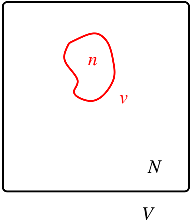
$N = V\rho$

n *particles in* v

n particles in volume v

Density fluctuations at equilibrium

Starting from $S = k \log \Omega$ Einstein



N particles

$N = V\rho$

n particles in v

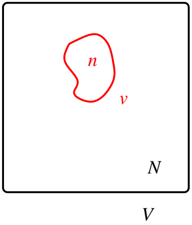
n particles in volume v

$$\langle n^2 \rangle - \langle n \rangle^2 = kT \; v\rho \; \kappa$$

κ is the compressibility,
ρ the density in the reservoir
and T the temperature.

Density fluctuations at equilibrium

Starting from $S = k \log \Omega$ Einstein



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κ is the compressibility,
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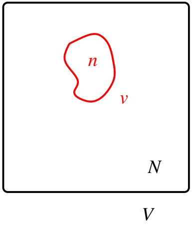
and T the temperature.

Variance of the fluctuation $= k \times$ Response coefficient

Boltzmann Constant $k \sim 10^{-23} J/K$

Large deviations of the density

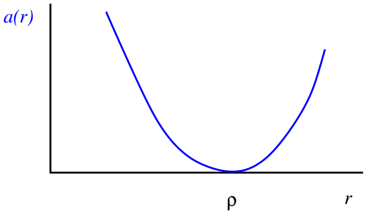


N particles

$N = V\rho$

n particles in v

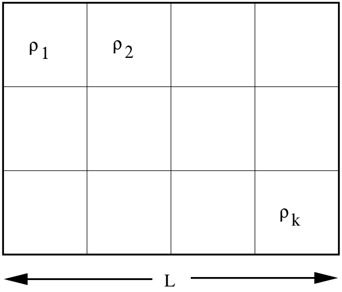
$$\text{Pro}\left(\frac{n}{v} = r\right) \sim \exp[-v\ a(r)]$$



$a(r)$ is the large deviation function

At equilibrium $a(r)$ is the free energy

Large deviation functional



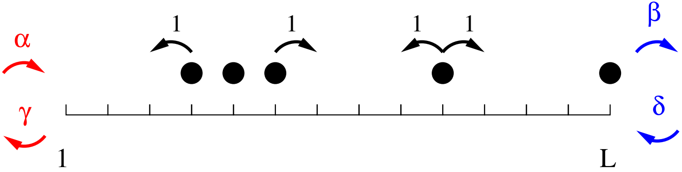
$$\text{Pro}(\rho_1, \ldots \rho_k) \sim \exp\left[-L^d \mathcal{F}(\rho_1, \ldots \rho_k)\right]$$

Large number k of boxes $\vec{r} = L\vec{x}$

$$\boxed{\text{Pro}(\{\rho(\vec{x})\}) \sim \exp\left[-L^d \mathcal{F}(\{\rho(\vec{x})\})\right]}$$

Exclusion processes

SSEP (Symmetric simple exclusion process)



$$\rho_a = \frac{\alpha}{\alpha + \gamma} \,, \qquad \rho_b = \frac{\delta}{\beta + \delta}$$

Large deviation function for the SSEP

$$\text{Pro}(\{\rho(x)\}) \sim \exp[-L\mathcal{F}(\{\rho(x)\})]$$

Equilibrium $\rho_a = \rho_b = F$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F} + \rho(x) \log \frac{\rho(x)}{F} \right]$$

Large deviation function for the SSEP

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Non-equilibrium ($\rho_a \neq \rho_b$)

D Lebowitz Speer 2001-2002
Bertini De Sole Gabrielli Jona-Lasinio Landim 2002

$$\mathcal{F} = \sup_{F(x)} \int dx \left[(1-\rho(x))\log\frac{1-\rho(x)}{1-F(x)} + \rho(x)\log\frac{\rho(x)}{F(x)} + \log\frac{F'(x)}{\rho_b - \rho_a}\right]$$

with $F(x)$ monotone, $F(0) = \rho_a$ and $F(1) = \rho_b$

Non-equilibrium $\rho_a \neq \rho_b$

$$\mathcal{F} = \sup_{F(x)} \int dx \left[(1-\rho(x)) \log \frac{1-\rho(x)}{1-F(x)} + \rho(x) \log \frac{\rho(x)}{F(x)} + \log \frac{F'(x)}{\rho_b - \rho_a} \right]$$

$$\Downarrow$$

F is non-local: for example for small $\rho_a - \rho_b$

$$\mathcal{F}(\{\rho(x)\}) = \int_0^1 dx \ (1-\rho(x)) \log \frac{1-\rho(x)}{1-\rho^*(x)} + \rho(x) \log \frac{\rho(x)}{\rho^*(x)}$$
$$+ \frac{(\rho_a - \rho_b)^2}{(\rho_a - \rho_a^2)^2} \int_0^1 dx \int_x^1 dy \ x(1-y) \big(\rho(x) - \rho^*(x)\big) \big(\rho(y) - \rho^*(y)\big)$$
$$+ O(\rho_a - \rho_b)^3$$

where $\rho^*(x) = \langle \rho(x) \rangle = (1-x)\rho_a + x\rho_b$

Long-range correlations Spohn 82

$$\langle \rho(x)\rho(y) \rangle - \langle \rho(x) \rangle \langle \rho(y) \rangle \simeq \frac{1}{L} G(x,y) = -\frac{(\rho_a - \rho_b)^2}{L} x(1-y)$$

Large deviation functional

$$\mathrm{Pro}(\{\rho(x)\}) \sim \exp[-\mathrm{Action}] = \exp\left[-L\mathcal{F}(\{\rho(x)\})\right]$$

Equilibrium

1. $\mathcal{F}$ local
2. $\mathcal{F} = T^{-1}\int d\vec{x}\, f(\rho(\vec{x}))$ f is the free energy per unit volume
3. Short range correlations

Large deviation functional

$$\mathrm{Pro}(\{\rho(x)\}) \sim \exp[-\mathrm{Action}] = \exp\left[-L\mathcal{F}(\{\rho(x)\})\right]$$

Equilibrium

1. $\mathcal{F}$ local
2. $\mathcal{F} = T^{-1}\int d\vec{x}\, f(\rho(\vec{x}))$ $\quad$ f is the free energy per unit volume
3. Short range correlations

Non-equilibrium

1. $\mathcal{F}$ non local
2. Weak long range correlations (SSEP for $x < y$) $\quad$ Spohn 1982

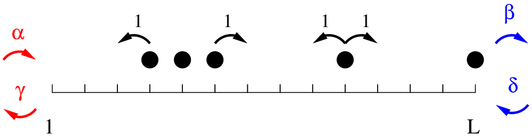
$$\langle(\rho(x) - \rho^*(x))(\rho(y) - \rho^*(y))\rangle \simeq \frac{1}{L}G(x,y) = -\frac{(\rho_a - \rho_b)^2}{L}x(1-y)$$

3. Higher correlations

$$\langle\rho(x_1)\rho(x_2))...\rho(x_n)\rangle_c \sim L^{1-n}$$

TWO APPROACHES

SSEP (Symmetric simple exclusion process)



Microscopic

Bethe ansatz, Perturbation theory, Computer,...

Macroscopic

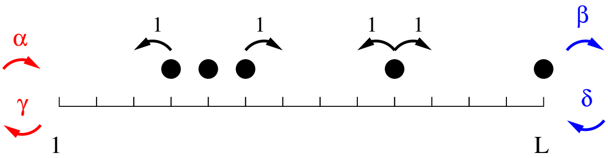
$i = Lx \ , \ t = L^2 \tau$

$$\mathrm{Pro}(\{\rho(x,\tau), j(x,\tau)\}) \sim \exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \, \frac{[j+\rho']^2}{4\rho(1-\rho)}\right]$$

Matrix ansatz

Fadeev 1980, ...,
D Evans Hakim Pasquier 1993

Steady state of the SSEP



$$P(\tau_1, \ldots, \tau_L) = \frac{\langle W|X_1 \ldots X_L|V\rangle}{\langle W|(D+E)^L|V\rangle}$$

where $X_i = \begin{cases} D & \text{if site } i \text{ occupied} \\ E & \text{if site } i \text{ empty} \end{cases}$

Matrix ansatz

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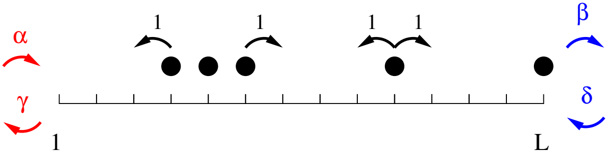
Steady state of the SSEP

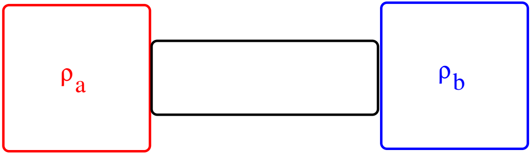


$$P(\tau_1, \dots, \tau_L) = \frac{\langle W | X_1 \dots X_L | V \rangle}{\langle W | (D+E)^L | V \rangle}$$

where $X_i = \begin{cases} D & \text{if site } i \text{ occupied} \\ E & \text{if site } i \text{ empty} \end{cases}$

$$\langle W | (\alpha E - \gamma D) = \langle W |$$
$$DE - ED = D + E$$
$$(\beta D - \delta E) | V \rangle = | V \rangle$$

Large diffusive systems

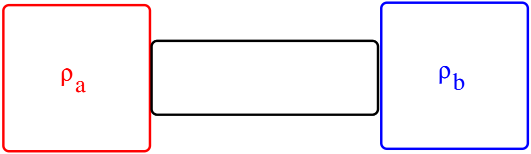


A diffusive system

- For $\rho_a - \rho_b$ small: $\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)\ (\rho_a - \rho_b)}{L}$ Fick's law

- $\rho_a = \rho_b = \rho$: $\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$

Large diffusive systems



A diffusive system

- For $\rho_a - \rho_b$ small: $$\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)\ (\rho_a - \rho_b)}{L}$$ Fick's law
- $\rho_a = \rho_b = \rho$: $$\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$$

Note that $\boxed{\sigma(\rho) = 2kT\rho^2\kappa(\rho)D(\rho)}$ where κ is the compressibility

Macroscopic fluctuation theory

Onsager Machlup theory
for non equilibrium diffusive systems

Kipnis Olla Varadhan 89
Spohn 91
⋮
Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001 $\to$

Evolution of a profile $\rho(x,t)$ for $0 \le t \le T$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\})$$
$$\exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \, \frac{[j(x,t)+\rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

- $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation law)
- $\rho(0,t) = \rho_a$; $\rho(1,t) = \rho_b$

Macroscopic fluctuation theory

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\})$$

$$\exp\left[-L\int_0^{T/L^2} dt \int_0^1 dx \; \frac{[j(x,t)+\rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t)D(\rho(x,t)) + \frac{1}{\sqrt{L}}\eta(x,t)$$

Macroscopic fluctuation theory

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\})$$
$$\exp\left[-L\int_0^{T/L^2} dt \int_0^1 dx \, \frac{[j(x,t)+\rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

$$\Updownarrow$$

$$j(x,t) = -\rho'(x,t)D(\rho(x,t)) + \frac{1}{\sqrt{L}}\eta(x,t)$$

with the white noise

$$\langle \eta(x,t)\eta(x',t')\rangle = 2\sigma(\rho(x,t))\delta(x-x')\delta(t-t')$$

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad ; \quad \rho(0,t) = \rho_a \; ; \; \rho(1,t) = \rho_b$$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\}) \sim \exp\left[-\mathrm{Action}\right] =$$

$$\exp\left[-L\int_0^{T/L^2} dt \int_0^1 dx \, \frac{[j(x,t) + \rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

How does a fluctuation $\rho(x,0)$ relax?

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad ; \quad \rho(0,t) = \rho_a \; ; \; \rho(1,t) = \rho_b$$

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How does a fluctuation $\rho(x,0)$ relax?

$$\mathrm{Action} = 0 \quad \Leftrightarrow \quad \frac{d\rho}{dt} = (D(\rho(x,t))\rho(x,t)')'$$

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad ; \quad \rho(0,t) = \rho_a \ ; \ \rho(1,t) = \rho_b$$

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How does a fluctuation $\rho(x,0)$ relax?

$$\mathrm{Action} = 0 \quad \Leftrightarrow \quad \frac{d\rho}{dt} = (D(\rho(x,t))\rho(x,t)')'$$

How is a fluctuation $\rho(x,0)$ produced?

$$\frac{d\rho}{dt} = -\frac{dj}{dx} \quad ; \quad \rho(0,t) = \rho_a \ ; \ \rho(1,t) = \rho_b$$

$$\mathrm{Pro}(\{\rho(x,t), j(x,t)\}) \sim \exp\left[-\mathrm{Action}\right] =$$

$$\exp\left[-L \int_0^{T/L^2} dt \int_0^1 dx \ \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}\right]$$

How does a fluctuation $\rho(x,0)$ relax?

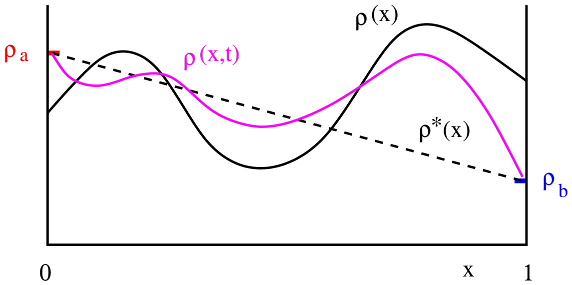
$$\mathrm{Action} = 0 \quad \Leftrightarrow \quad \frac{d\rho}{dt} = (D(\rho(x,t))\rho(x,t)')'$$

How is a fluctuation $\rho(x,0)$ produced?

Minimize the Action with $\rho(x,-\infty) = \rho_{\mathrm{steady\ state}}(x)$

Macroscopic fluctuation theory

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2001-2002



$$\mathcal{F}(\{\rho(x)\}) = \min_{\rho(x,t)} \int_{-\infty}^{0} dt \int_0^1 dx \, \frac{[j(x,t) + \rho'(x,t) D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\rho(x,0) = \rho(x)$; $\rho(x,-\infty) = \rho^*(x)$

SSEP

How does a fluctuation $\rho(x, 0)$ relax?

$$\frac{d\rho(x,t)}{dt} = \frac{d^2\rho(x,t)}{dx^2}$$

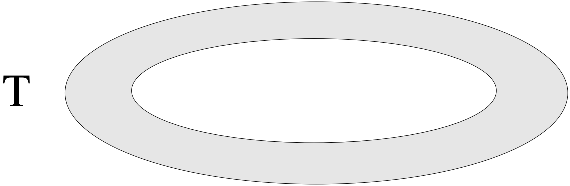
How is a fluctuation $\rho(x, 0)$ produced?

$$\frac{d\rho(x,s)}{ds} = \frac{d^2\rho(x,s)}{dx^2} - 2\frac{(\rho_a - \rho_b)}{\rho_a(1-\rho_a)}(1 - 2\rho(x,s))\frac{d\rho(x,s)}{dx} + O\left((\rho_a - \rho_b)^2\right)$$

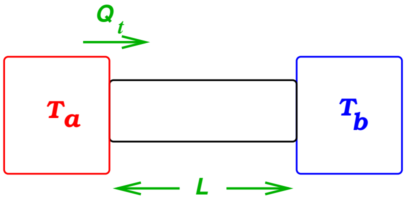
Current fluctuations in non-equilibrium steady states

Current fluctuations in non-equilibrium steady states

Current fluctuations on a ring

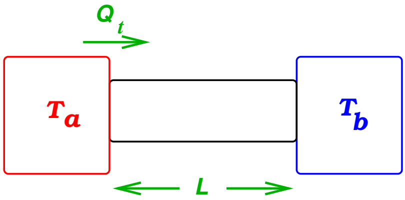


AVERAGE CURRENT OF **HEAT**



$$\frac{\langle Q_t \rangle}{t} = F(T_a, T_b, L, \text{contacts})$$

AVERAGE CURRENT OF **HEAT**



$$\frac{\langle Q_t \rangle}{t} = F(T_a, T_b, L, \text{contacts})$$

FOURIER's law:

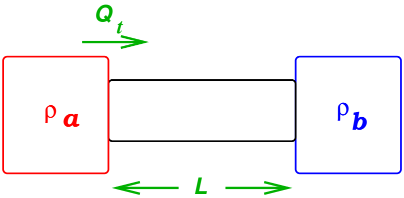
large L

$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(T_a, T_b)}{L}$$

large L and $T_a - T_b$ small

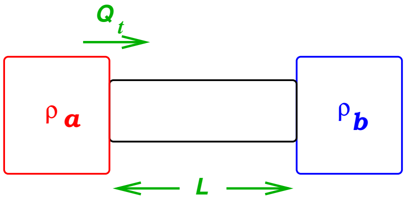
$$\frac{\langle Q_t \rangle}{t} \simeq D(T)\frac{T_a - T_b}{L} \simeq -D(T)\nabla T$$

AVERAGE CURRENT OF **PARTICLES**



$$\frac{\langle Q_t \rangle}{t} = F(\rho_a, \rho_b, L, \text{contacts})$$

AVERAGE CURRENT OF **PARTICLES**



$$\frac{\langle Q_t \rangle}{t} = F(\rho_a, \rho_b, L, \mathrm{contacts})$$

FICK's law:

large L

$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(\rho_a, \rho_b)}{L}$$

large L and $\rho_a - \rho_b$ small

$$\frac{\langle Q_t \rangle}{t} \simeq D(\rho)\frac{\rho_a - \rho_b}{L} \simeq -D(\rho)\nabla\rho$$

ONE DIMENSIONAL MECHANICAL SYSTEMS

Ideal gas

No Fourier's law

Harmonic chain

$$E = \sum \frac{p_i^2}{2m} + \sum_i g(x_{i+1} - x_i)^2$$

$$\frac{\langle Q_t \rangle}{t} \simeq G(T_a, T_b)$$

ONE DIMENSIONAL MECHANICAL SYSTEMS

Ideal gas

No Fourier's law

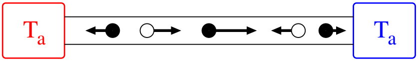
Harmonic chain

$$E = \sum \frac{p_i^2}{2m} + \sum_i g(x_{i+1} - x_i)^2$$

$$\frac{\langle Q_t \rangle}{t} \simeq G(T_a, T_b)$$

Hard particle gas

Anomalous Fourier's law



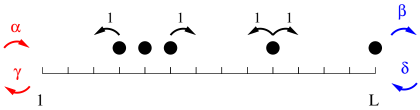
$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(T_a, T_b)}{L^{1-\alpha}}$$

$.3 \leq \alpha \leq .5$ Delfini Lepri Livi Politi Livi, Spohn, Grassberger, ...

Anharmonic chain

DIFFUSIVE SYSTEMS: average current

Symmetric exclusion



Fourier's law

$$\frac{\langle Q_t \rangle}{t} \simeq \frac{G(T_a, T_b)}{L}$$

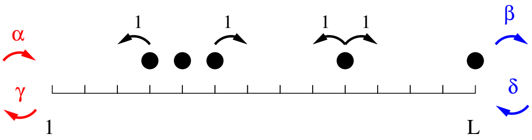
General lattice gas

Random walkers

KMP model Kipnis Marchioro Presutti

SSEP (Symmetric simple exclusion process)

D. Douçot Roche 2004



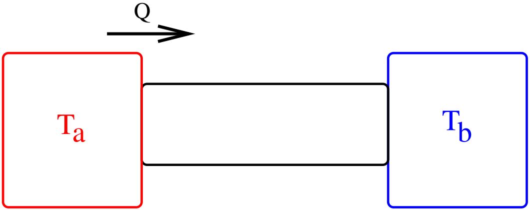
$$\rho_a = \frac{\alpha}{\alpha + \gamma} \,, \qquad \rho_b = \frac{\delta}{\beta + \delta}$$

$$\lim_{t \to \infty} \frac{\langle Q(t) \rangle}{t} \simeq \frac{1}{L} [\rho_a - \rho_b]$$ Fick's law

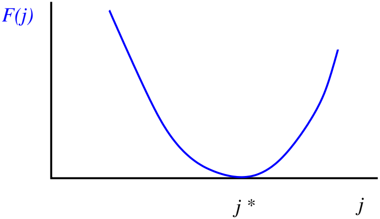
$$\lim_{t \to \infty} \frac{\langle Q^2(t) \rangle_c}{t} \simeq \frac{1}{L} \left[\rho_a + \rho_b - \frac{2(\rho_a^2 + \rho_a \rho_b + \rho_b^2)}{3} \right]$$

$$\lim_{t \to \infty} \frac{\langle Q^3(t) \rangle_c}{t} \simeq \frac{1}{L} (\rho_a - \rho_b) \left[1 - 2(\rho_a + \rho_b) + \frac{16\rho_a^2 + 28\rho_a \rho_b + 16\rho_b^2}{15} \right]$$

CURRENT FLUCTUATIONS AND LARGE DEVIATIONS



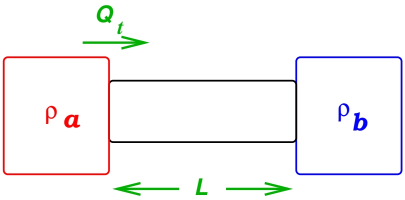
Q_t Energy transferred during time t



$$\text{Pro}\left(\frac{Q_t}{t} = j\right) \sim \exp[-t\ F(j)]$$

Expansion of $F(j)$ near j^* gives all cumulants of Q_t

DIFFUSIVE SYSTEMS: higher cumulants



Bodineau D. 2004

- For $\rho_a - \rho_b$ small: $$\frac{\langle Q_t \rangle}{t} = \frac{D(\rho)(\rho_a - \rho_b)}{L}$$
- $\rho_a = \rho_b = \rho$: $$\frac{\langle Q_t^2 \rangle}{t} = \frac{\sigma(\rho)}{L}$$

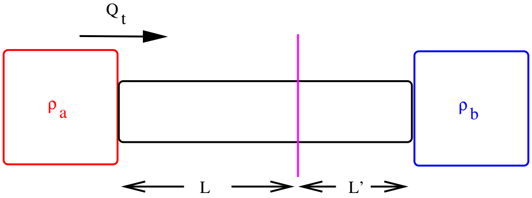
Then one can calculate

All cumulants of Q_t for arbitrary ρ_a and ρ_b

ADDITIVITY PRINCIPLE

$$\text{Pro}\left(\frac{Q_t}{t} = j, \rho_a, \rho_b\right) \sim \exp[-t \ F_{L+L'}(j, \rho_a, \rho_b)]$$

Bodineau D. 2004



Additivity

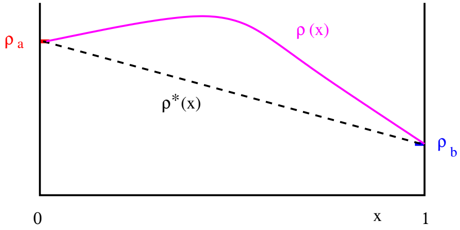
$$P_{L+L'}(Q, \rho_a, \rho_b) \sim \max_{\rho} [P_L(Q, \rho_a, \rho) \ P_{L'}(Q, \rho, \rho_b)]$$

$$F_{L+L'}(j, \rho_a, \rho_b) = \min_{\rho} [F_L(j, \rho_a, \rho) \ + \ F_{L'}(j, \rho, \rho_b)]$$

Bodineau D. 2004

VARIATIONAL PRINCIPLE

$$F_L(j, \rho_a, \rho_b) = \frac{1}{L} \min_{\rho(x)} \int_0^1 dx \, \frac{[j \, L \; + \rho'(x) \; D(\rho(x))]^2}{2\sigma(\rho(x))}$$



Satisfies the fluctuation theorem

$$F(j) - F(-j) = j \int_{\rho_a}^{\rho_b} \frac{D(\rho)}{\sigma(\rho)} d\rho$$

Gallavotti Cohen 1995
Evans Searls 1994
.... Kurchan 1998
Lebowitz Spohn 1999

DIFFUSIVE SYSTEMS: all the cumulants

Bodineau D. 2004

$$\frac{\langle Q_t \rangle_c}{t} = \frac{1}{L} \mathcal{I}_1$$

$$\frac{\langle Q_t^2 \rangle_c}{t} = \frac{1}{L} \frac{\mathcal{I}_2}{\mathcal{I}_1}$$

$$\frac{\langle Q_t^3 \rangle_c}{t} = \frac{1}{L} \frac{3\,(\,\mathcal{I}_3\,\mathcal{I}_1 - \mathcal{I}_2^2\,)}{\mathcal{I}_1^3}$$

$$\frac{\langle Q_t^4 \rangle_c}{t} = \frac{1}{L} \frac{3\,(\,5\,\mathcal{I}_4\,\mathcal{I}_1^2 - 14\,\mathcal{I}_1 \mathcal{I}_2\,\mathcal{I}_3 + \,9\,\mathcal{I}_2^3\,)}{\mathcal{I}_1^5}$$

where

$$\mathcal{I}_n = \int_{\rho_b}^{\rho_a} D(\rho)\sigma(\rho)^{n-1} d\rho$$

For the SSEP $D(\rho) = 1$ and $\sigma(\rho) = 2\rho(1-\rho)$
For the KMP $D(\rho) = 1$ and $\sigma(\rho) = 4\rho^2$
For the random walkers $D(\rho) = 1$ and $\sigma(\rho) = 2\rho$

TRUE VARIATIONAL PRINCIPLE

Bertini De Sole Gabrielli
Jona-Lasinio Landim 2005

$$F(j) = \frac{1}{L} \lim_{T\to\infty} \min_{\rho(x,t),j(x,t)} \frac{1}{T} \int_0^T dt \int_0^1 dx \frac{[j(x,t) + \rho'(x,t)D(\rho(x,t))]^2}{2\sigma(\rho(x,t))}$$

with $\frac{d\rho}{dt} = -\frac{dj}{dx}$ (conservation), $\rho_t(0) = \rho_a$, $\rho_t(1) = \rho_b$ and

$$j\ T = \int_0^T j_t(x) dt$$

- Sufficient condition for the optimal profile to be time independent
- Dynamical phase transition

Bodineau D. 2005-2007

the optimal $\rho_t(x)$ starts to become time dependent

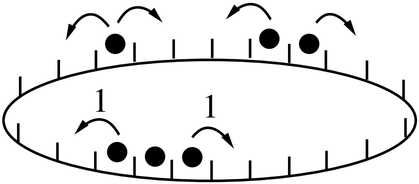
SSEP ON A RING

Appert D Lecomte Van Wijland 2008

N particles

L sites

$$\rho = \frac{N}{L}$$



Q_t flux through a bond during time t

CUMULANTS OF THE CURRENT FOR THE SSEP ON A RING N particles and L sites

$$\sigma(\rho) = 2\rho(1-\rho) = \frac{2N(L-N)}{L^2}$$

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L-1}$$

$$\frac{\langle Q^4 \rangle_c}{t} = \frac{\sigma^2}{2(L-1)^2}$$

$$\frac{\langle Q^6 \rangle_c}{t} = -\frac{(L^2-L+2)\sigma^3 - 2(L-1)\sigma^2}{4(L-1)^3(L-2)}$$

$$\frac{\langle Q^8 \rangle_c}{t} = \frac{(10L^4-2L^3+27L^2-15L+18)\sigma^4 - 4(L-1)(11L^2-L+12)\sigma^3 + 48(L-1)^2\sigma^2}{24(L-1)^4(L-2)(L-3)}$$

$$\boxed{\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}}$$ Gaussian + Fick's law

$$\boxed{\frac{\langle Q^{2n} \rangle_c}{t} \sim \frac{\sigma^n}{L^2}}$$ for $n \geq 2$

UNIVERSAL CUMULANTS OF THE CURRENT

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

UNIVERSAL CUMULANTS OF THE CURRENT

$$\boxed{\frac{\langle Q^2\rangle_c}{t} = \frac{\sigma}{L}}$$

Gaussian

$$\boxed{\frac{\langle Q^4\rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6\rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8\rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}}$$

Universal

Macroscopic fluctuation theory

UNIVERSAL CUMULANTS OF THE CURRENT

$$\frac{\langle Q^2 \rangle_c}{t} = \frac{\sigma}{L}$$

Gaussian

$$\frac{\langle Q^4 \rangle_c}{t} \simeq \frac{\sigma^2}{2L^2}, \quad \frac{\langle Q^6 \rangle_c}{t} \simeq -\frac{\sigma^3}{4L^2}, \quad \frac{\langle Q^8 \rangle_c}{t} \simeq \frac{5\sigma^4}{12L^2}$$

Universal

Macroscopic fluctuation theory

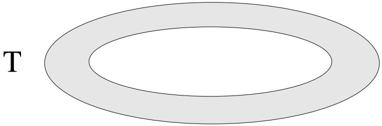
+ fluctuations around a flat profile

DIFFUSIVE SYSTEMS: summary

Open system

$$\langle Q^n \rangle_c / t \sim L^{-1}$$

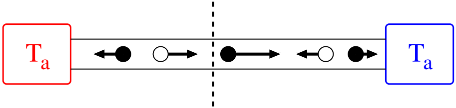
Ring



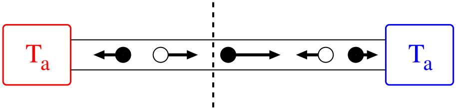
$$\langle Q^2 \rangle_c / t \sim L^{-1}$$

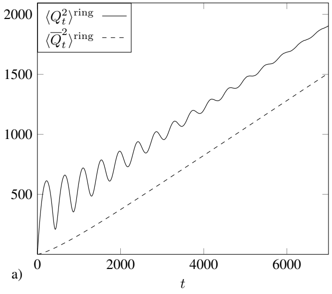
$$\langle Q^{2n} \rangle_c / t \sim L^{-2} \quad \text{for} \quad n \geq 2$$

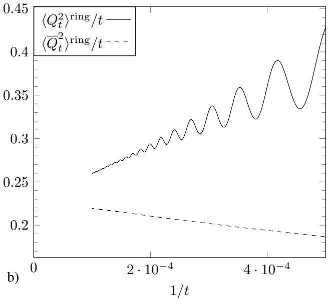
HARD PARTICLE GAS: the second cumulant



HARD PARTICLE GAS: the second cumulant



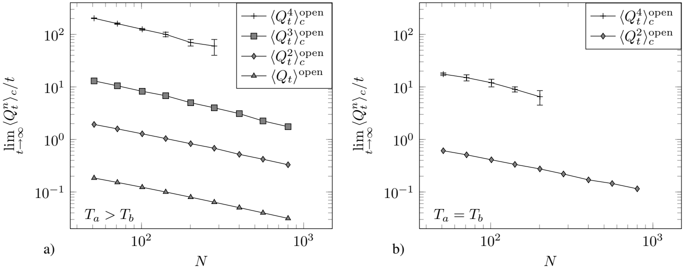




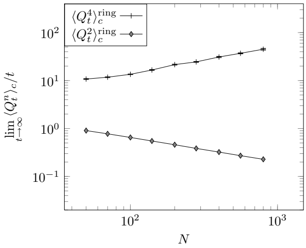
$\frac{\langle Q^2 \rangle}{t}$ has a limit

HARD PARTICLE GAS: Size dependence of the cumulants

OPEN SYSTEM

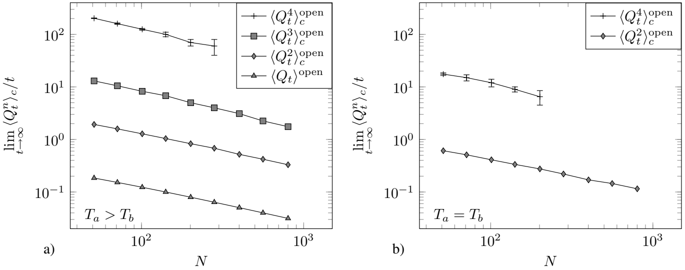


RING

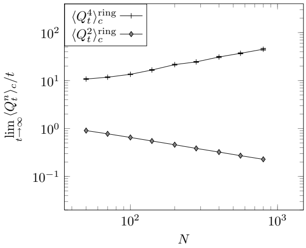


HARD PARTICLE GAS: Size dependence of the cumulants

OPEN SYSTEM

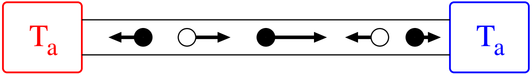


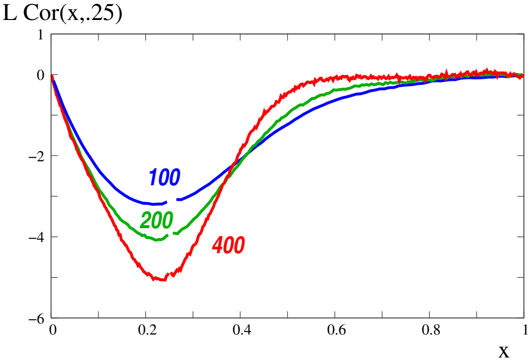
RING



ANOMALOUS FOURIER'S LAW

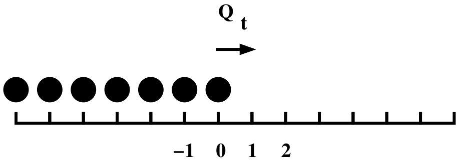
Correlations on a mesoscopic scale





Delfini, Lepri, Livi, Mejia-Monasterio, Politi 2010
Gerschenfeld, D., Lebowitz 2010

Step initial condition



ASEP

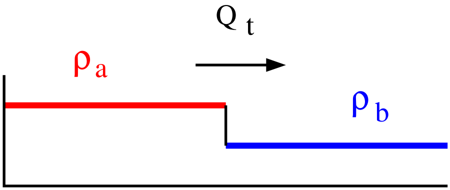
G. Schütz 98

⋮

Prähofer Spohn 2000-2002

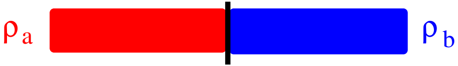
⋮

Tracy Widom 2008



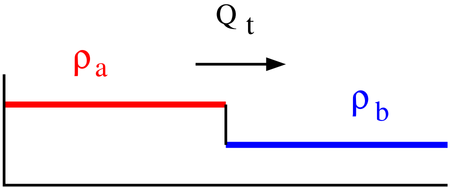
SSEP

D Gerschenfeld 2009



$$\left\langle e^{\lambda Q_t} \right\rangle = ?$$

Step initial condition on the infinite line



SSEP

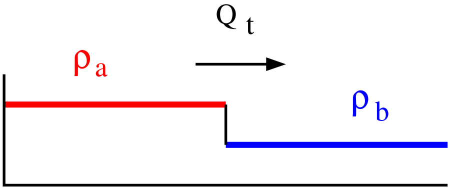
D Gerschenfeld 2009

$$\left\langle e^{\lambda Q_t} \right\rangle \simeq \exp[\sqrt{t}\ H(\omega)]$$

with $H(\omega) = \frac{1}{\pi} \int_{\infty}^{\infty} dk \log \left[1 + \omega e^{-k^2} \right]$

and $\omega = 1 - [1 - (e^{\lambda} - 1)\rho_a][1 - (1 - e^{-\lambda})\rho_b]$

Step initial condition on the infinite line



SSEP

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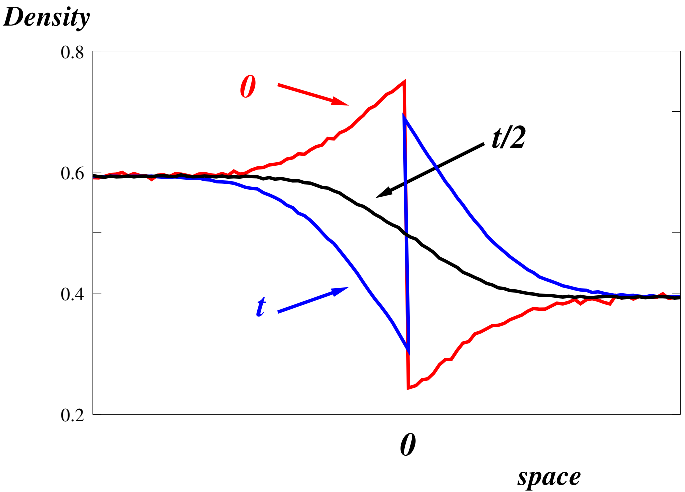
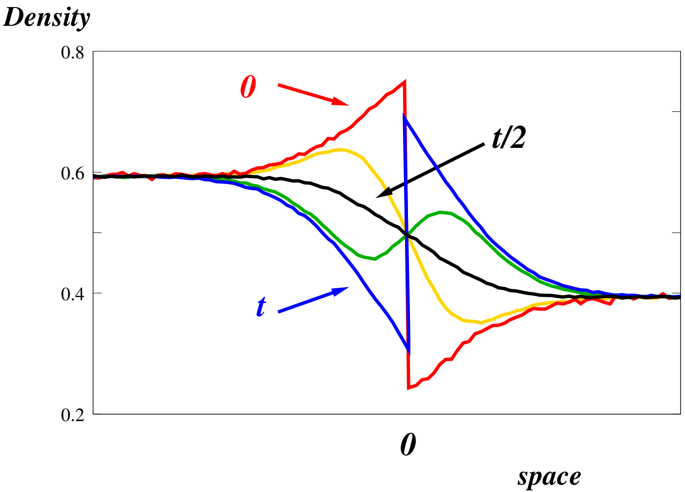
and $\omega = 1 - [1 - (e^{\lambda} - 1)\rho_a][1 - (1 - e^{-\lambda})\rho_b]$

For large Q_t : $\mathrm{Pro}(Q_t) \sim \exp\left[-\frac{\pi^2}{12}\ Q_t^3/t\right]$

Density profiles condionned on the current



Density profiles condionned on the current



OPEN QUESTIONS

Large deviation function of the density
for a general diffusive system

Non steady state situations

Theory for mechanical systems which conserved impulsion

Mean field theory for large deviation functions