Extreme values: a renormalization group approach

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Many extreme value problems...

- Reaction path with the lowest barrier in a complex landscape
- Ground state in a disordered system
- Problems of pinned interfaces,...

In many cases, one needs to find minimum or maximum values among a set of random variables $\Rightarrow$ *statistics?*

See, e.g., Bouchaud, Mézard, J. Phys. A (1997).

Difficulties

- Presence of strong correlations, multiples scales,...
- Use of renormalization group could be relevant (but still difficult)
- What about the simplest extreme value problem, with iid random variables?

Standard results

- Variables $x_1, \ldots, x_n$ drawn from cumulative distribution $\mu(x)$ (called parent distribution)
- Rescaled cumulative distribution of $\max(x_1, \ldots, x_n)$

$$\mathcal{F}_\gamma(y) = exp[-(1+\gamma y)]^{-1/\gamma} \qquad 1+\gamma y > 0$$

$\gamma > 0$: Fréchet distribution (power-law tail of parent dist.)

$\gamma = 0$: Gumbel distribution (faster than powel-law tail)

$\gamma < 0$: Weibull distribution (bounded variables)

Fisher, Tippett (1928); Gnedenko (1943); Gumbel (1958)

Motivation

- Asymptotic distributions of extreme values of iid random variables known for long, but strong finite-size effects, not always easy to handle with standard probabilistic methods
- Idea: Use the renormalization language as a convenient tool to analyze fixed points and finite-size corrections, in spite of the absence of correlations
- Approach initiated in

  Györgyi, Moloney, Ozogány, Rácz, PRL (2008)

  Györgyi, Moloney, Ozogány, Rácz, Droz, PRE (2010)
- Aim of the present contribution: reformulate the results using a differential representation, which is more convenient

Basic renormalization idea

Extreme value statistics

- N iid random variables, cumulative distribution $\mu(x) = \int_{-\infty}^{x} \rho(x')dx'$
- Cumulative distribution for the maximum value

$$\mathrm{Prob}(\max(x_1, \ldots, x_N) < x) = \mathrm{Prob}(\forall i, x_i < x) = \mu^N(x)$$

Decimation procedure

- Split the set of sufficiently large N random variables x_i into $N' = N/p$ blocks of p random variables each
- y_j the maximum value in the j^{th} block

$$\max(x_1, \ldots, x_N) = \max(y_1, \ldots, y_{N'})$$

- y_j are also i.i.d. random variables, with a distribution $\mu_p(y)$

$$\mu_p(y) = \mu^p(y)$$

Renormalization operation

Raising to a power and rescaling

$$[\hat{R}_p \mu](x) = \mu^p\left(a_p x + b_p\right)$$

- Necessity of scale and shift parameters a_p and b_p to lift degeneracy of the distribution
- Conditions to fix a_p and b_p to be specified later on

Parameterization of the flow

- p considered as continuous rather than discrete
- change of flow parameter $p = e^s$: distribution $\mu(x,s)$, parameters $a(s)$ and $b(s)$
- Parent distribution $\mu(x)$ obtained for $s = 0$

$$\mu(x,0) = \mu(x)$$

Renormalization operation

Change of function

- double exponential form

$$\mu(x,s) = e^{-e^{-g(x,s)}}$$

- Link to the parent distribution: $g(x, s=0) = g(x)$

Standardization conditions

- Conditions to fix the parameters $a(s)$ and $b(s)$

$$\mu(0,s) \equiv e^{-1}, \qquad \partial_x \mu(0,s) \equiv e^{-1}$$

- In terms of the function $g(x,s)$

$$g(0,s) \equiv 0, \qquad \partial_x g(0,s) \equiv 1$$

Renormalization operation

Renormalization of $\mu(x, s)$

$$\mu(x, s) \equiv [\hat{R}_s \mu](x) = \mu^{e^s}\big(a(s)x + b(s)\big)$$

Renormalization of $g(x, s) = -\ln[-\ln \mu(x, s)]$

$$g(x, s) = g\big(a(s)x + b(s)\big) - s.$$

Very simple transformation: linear change of variable in the argument and global additive shift.

However, one needs to determine $a(s)$ and $b(s)$.

Differential representation

Iteration of the RG transformation

$$g(x, s + \Delta s) = [\hat{R}_{\Delta s} g](x, s)$$

Infinitesimal transformation $\Delta s = ds$

$$g(x, s + ds) = [\hat{R}_{ds} g](x, s)$$

- More explicitly, with $a(ds) = 1 + \gamma(s)ds$ and $b(ds) = \eta(s)ds$:

$$g(x, s + ds) = g\Big(\big(1 + \gamma(s)ds\big)x + \eta(s)ds, s\Big) - ds$$

  where the functions $\gamma(s)$ and $\eta(s)$ are to be specified

- Linearizing with respect to ds, we get

$$\partial_s g(x, s) = \big(\gamma(s)x + \eta(s)\big)\partial_x g(x, s) - 1$$

Partial differential equation

Determination of $\gamma(s)$ and $\eta(s)$

- Standardiz. conditions $g(0,s) \equiv 0$ and $\partial_x g(0,s) \equiv 1$ yield

$$\eta(s) \equiv 1$$

$$\gamma(s) = -\partial_x^2 g(0,s)$$

Partial differential equation of the flow

$$\partial_s g(x,s) = (1 + \gamma(s)\, x)\, \partial_x g(x,s) - 1$$

Fixed points of the flow

- Stationary solution $g(x,s) = f(x)$:

$$0 = (1+\gamma x) f'(x) - 1 \qquad \text{with} \ \ \gamma = -f''(0)$$

- Using the standardization condition $f(0) = 0$

$$f(x;\gamma) = \int_0^x (1+\gamma y)^{-1} dy = \frac{1}{\gamma} \ln(1+\gamma x)$$

- Fixed point integrated distribution

$$M(x;\gamma) = e^{-e^{-f(x;\gamma)}} = e^{-(1+\gamma x)^{-1/\gamma}}$$

Easy way to recover the well-known generalized extreme value distributions, obtained here as a fixed line of the RG transformation

Linear perturbations

- Perturbation $\phi(x,s)$ introduced through

$$g(x,s) = f(x) + f'(x)\,\phi(x,s)$$

- Linearized partial differential equation

$$\partial_s \phi(x,s) = (1+\gamma x)\,\partial_x \phi(x,s) - \gamma\,\phi(x,s) - x\,\partial_x^2 \phi(0,s)$$

- Convergence properties to the fixed point distribution are obtained from the analysis of this PDE

Eigenfunctions

Perturbations of the form $\phi(x,s) = e^{\gamma' s}\,\psi(x)$

Solution for the Weibull and Fréchet cases ($\gamma \neq 0$)

$$\psi(x;\gamma,\gamma') = \frac{1 + (\gamma'+\gamma)x - (1+\gamma x)^{\gamma'/\gamma+1}}{\gamma'(\gamma'+\gamma)}$$

in the range of x such that $1 + \gamma x > 0$.

Solution for the Gumbel case ($\gamma = 0$)

$$\psi(x;\gamma') = \frac{1}{\gamma'^2}\left(1 + \gamma' x - e^{\gamma' x}\right)$$

Empirical interpretation

- N variables in the block $\Rightarrow s = \ln N$
- Convergence $g(x, s = \ln N) \to f(x)$
- Corrections proportional to $e^{\gamma' s} \propto N^{\gamma'}$
  (if $\gamma' = 0$: logarithmic convergence in N).
- Interpretation of $\gamma' > 0$? Are there unstable solutions?

  $\Rightarrow$ Can we look at non-perturbative solutions?

Motivation

Unstable solutions around the fixed point may seem counterintuitive: can we find an example of full RG trajectory starting from an unstable direction?

Back to the equations: the Gumbel case

- Equation to be solved

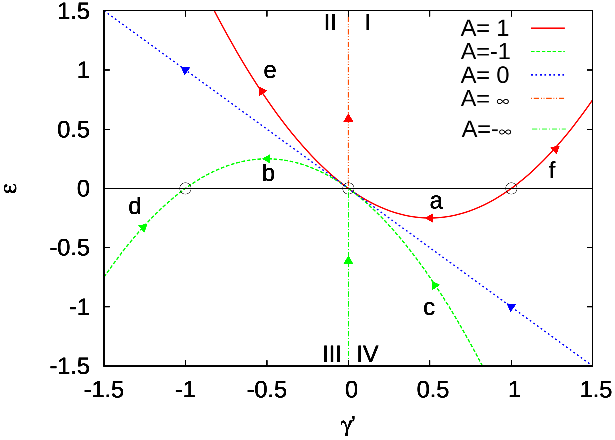
$$\partial_s g(x,s) = (1+\gamma(s)\,x)\,\partial_x g(x,s) - 1$$

- Ansatz for the solution starting from $f(x) = x$

$$g(x,s) = x + \epsilon(s)\,\psi\big(x;\gamma'(s)\big)$$

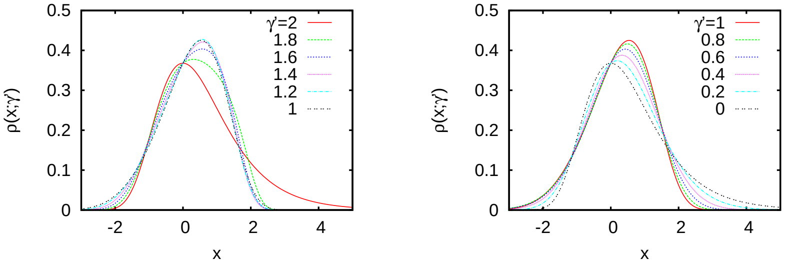
- Same as linear perturbation, except that γ' depends on s

Parameter space (ϵ, γ')



Evolution of the distributions

Starting close to the Gumbel distribution ($\gamma' = 2$)… and coming back to it (at $\gamma' = 0$) after an excursion



Bertin, Györgyi, J. Stat. Mech. (2010)

Generalization of standard extreme statistics

Raising the variables to an increasing power

- Choose iid variables whose statistics depends on the sample size n, for instance by raising $x_1, \ldots, x_n$ to a power q_n
- Question: statistics of the quantity $\max(x_1^{q_n}, \ldots, x_n^{q_n})$
- Motivation: link with the Random Energy Model

  Ben Arous et.al. (2005), Bogachev (2007)

Results

- Emergence of new limit distributions, for $q(n) \sim n^Q$

$$\mathcal{F}_{\gamma,Q} = \exp\left[-\left(1 - \frac{Q}{\gamma} \ln(1+\gamma x)\right)^{1/Q} \right]$$

- Standard distributions recovered for $Q \to 0$

Angeletti, Bertin, Abry, J. Phys. A (2012)

Formal analogy between sums and extremes

Extreme value statistics for iid random variables

- Relevant mathematical object: integrated distribution $\mu(x)$
- Integrated distribution of the maximum of N iid random variables

$$\mu_N(x) = \mu(x)^N$$

- Linear rescaling of x to preserve the standardiz. conditions

Statistics of sums of iid random variables

- Relevant mathematical object: characteristic function $\Phi(q)$
- Characteristic function for the sum of N iid random variables

$$\Phi_N(q) = \Phi(q)^N$$

- Linear rescaling

Same formal structure, only the objects differ

Result for the characteristic function

$$\Phi(q;\gamma) = e^{-|q|^{-\frac{1}{\gamma}}}$$

- Characteristic function of the symmetric Lévy distribution, of parameter $\alpha = -1/\gamma$.
- Here, one restriction: $\gamma \leq -\frac{1}{2}$, equivalent to $0 < \alpha \leq 2$
- Linear stability analysis (eigenfunctions, ...) can be performed in the same way as for extreme value statistics

Bertin, Györgyi, J. Stat. Mech. (2010)

Conclusion

On the present work

- Renormalization is a convenient tool to analyze fixed points and finite size corrections
- Analysis of finite size corrections made easy by the use of eigenfunctions
- Can be applied to variants of the present problems, for instance, statistics of $\max(x_1^{q_n}, \ldots, x_n^{q_n})$

Outlook

- Is renormalization without correlation really renormalization? Extension to correlated variables welcome... but yet unclear